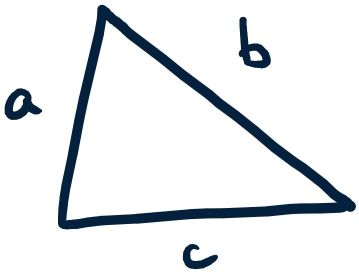
DESIGUALDADES TRIGONOMÉTRICAS

• DESIGUALDADE TRIANGULAR •



$$a + b > c$$

• DESIGUALDADE DAS MÉDIAS •

$$M.A. \geq M.G.$$

$$\frac{a+b}{2} \geq \sqrt{a \cdot b}$$

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0 \quad \therefore \quad a - 2\sqrt{a}\sqrt{b} + b \geq 0$$

$$a + b \geq 2\sqrt{a \cdot b} \longrightarrow \frac{a+b}{2} \geq \sqrt{a \cdot b}$$

EXEMPLO

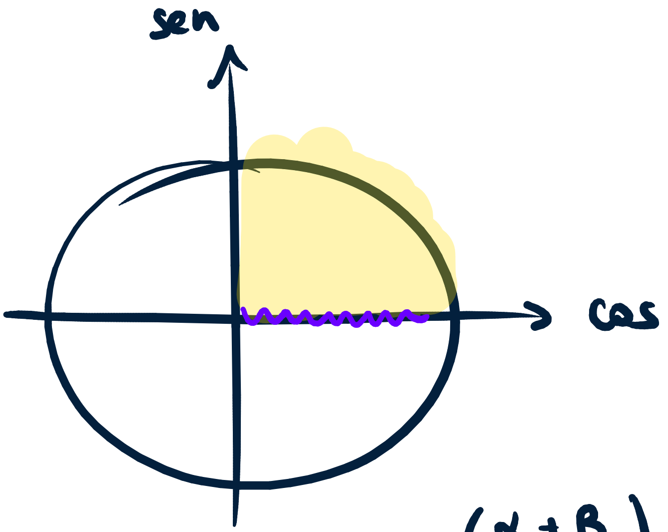
Mostre que $\operatorname{sen}\left(\frac{\alpha+\beta}{2}\right) \geq \frac{\operatorname{sen}\alpha + \operatorname{sen}\beta}{2}$ p/

$\alpha, \beta \in [0, \pi/2]$

R:

$$\operatorname{sen}\alpha + \operatorname{sen}\beta = 2\operatorname{sen}\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\operatorname{sen}\left(\frac{\alpha+\beta}{2}\right) = \frac{\operatorname{sen}\alpha + \operatorname{sen}\beta}{2} \cdot \underbrace{\boxed{\frac{1}{\cos\left(\frac{\alpha-\beta}{2}\right)}}}_{\geq 1}$$

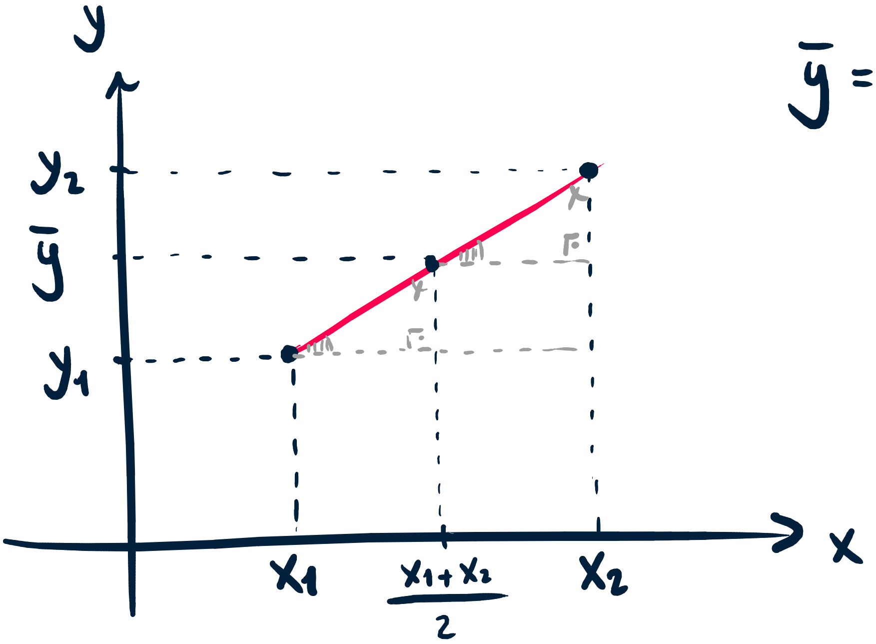


$$\operatorname{sen}\left(\frac{\alpha+\beta}{2}\right) \geq \frac{\operatorname{sen}\alpha + \operatorname{sen}\beta}{2}$$

UNIVERSO NARRADO

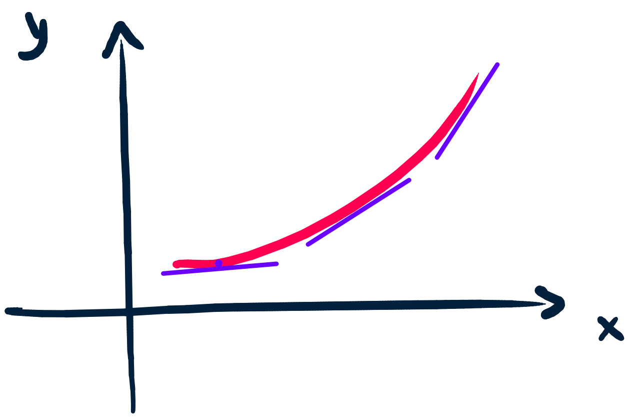
DESIGUALDADE DE JENSEN

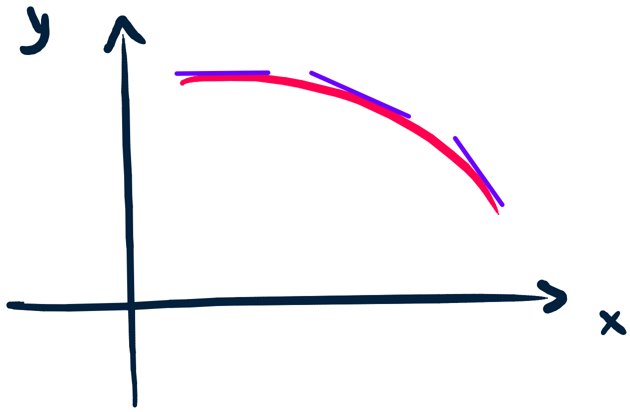
OBS.:

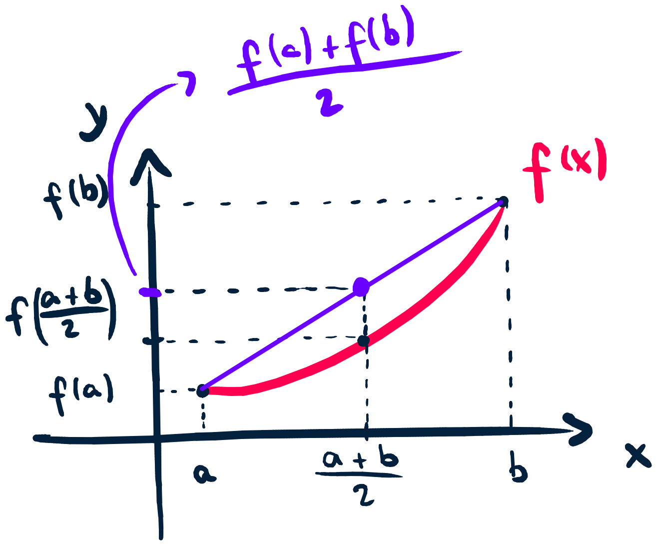


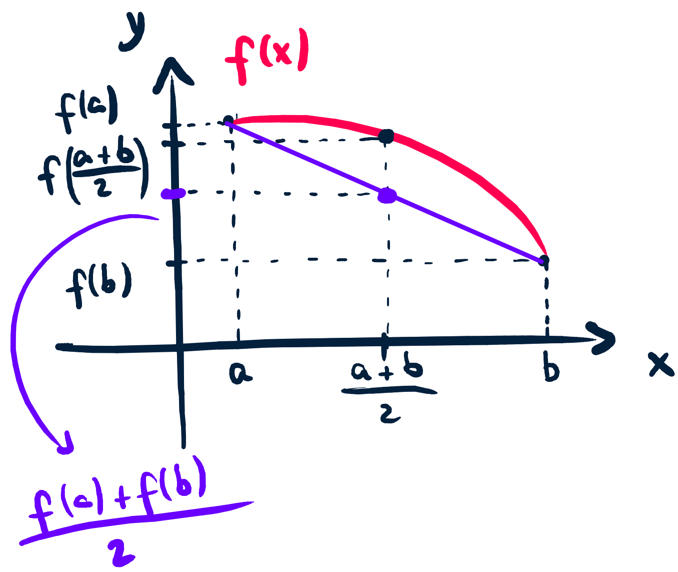
$$\bar{y} = \frac{y_1 + y_2}{2}$$

FUNÇÕES CONVEXAS vs FUNÇÕES CÔNCAVAS









$$\frac{f(a)+f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$$

$$\frac{f(a)+f(b)}{2} \leq f\left(\frac{a+b}{2}\right)$$

GENERALIZANDO :

FUNÇÕES CONVEXAS

$$\boxed{\frac{f(a_1)+f(a_2)+\dots+f(a_n)}{n} \geq f\left(\frac{a_1+a_2+\dots+a_n}{n}\right)}$$

FUNÇÕES CÔNCAVAS

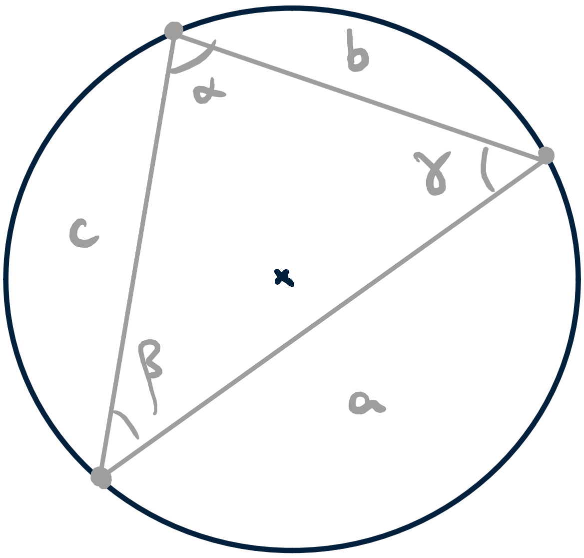
$$\boxed{\frac{f(a_1)+f(a_2)+\dots+f(a_n)}{n} \leq f\left(\frac{a_1+a_2+\dots+a_n}{n}\right)}$$

EXEMPLO 01

DADO UM CÍRCULO DE RAIO UNITÁRIO, QUAL É O TRIÂNGULO INSCRITO DE PERÍMETRO MÁXIMO?

R:

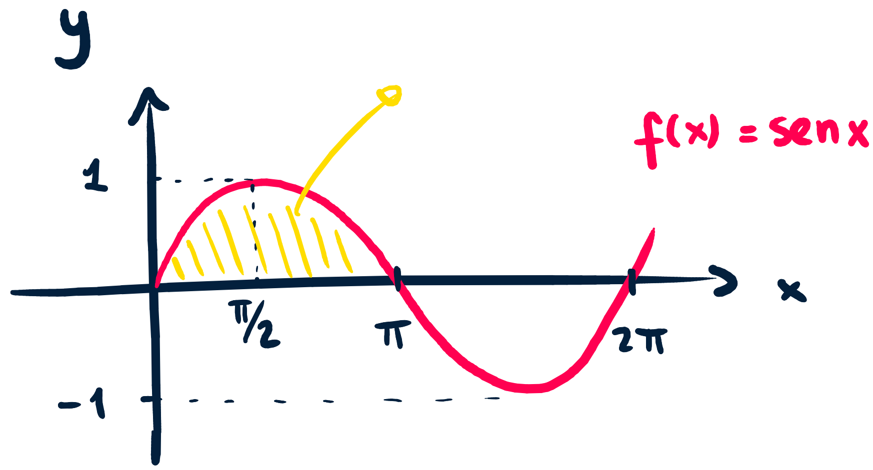
$$2p = a+b+c$$



LEI DOS SENOS:

$$\frac{a}{\operatorname{sen}\alpha} = \frac{b}{\operatorname{sen}\beta} = \frac{c}{\operatorname{sen}\gamma} = \overset{1}{2\cancel{R}} = \frac{a+b+c}{\operatorname{sen}\alpha + \operatorname{sen}\beta + \operatorname{sen}\gamma}$$

$$2p = a+b+c = 2.(\operatorname{sen}\alpha + \operatorname{sen}\beta + \operatorname{sen}\gamma)$$



$$\frac{f(\alpha)+f(\beta)+f(\gamma)}{3} \le f\left(\frac{\overbrace{\alpha+\beta+\gamma}^{\pi}}{3}\right)$$

$$\frac{\text{sen}\,\alpha+\text{sen}\,\beta+\text{sen}\,\gamma}{3} \le \text{sen}\left(\pi/3\right)$$

$$\text{sen}\,\alpha+\text{sen}\,\beta+\text{sen}\,\gamma \le 3\cdot\frac{\sqrt{3}}{2}$$

$$\alpha=\beta=\gamma=\frac{\pi}{3}$$

↳ TRIÂNGULO EQUILÁTERO

$$\left(\text{sen}\,\alpha+\text{sen}\,\beta+\text{sen}\,\gamma\right)_{\text{máx}} = 3\cdot\frac{\sqrt{3}}{2}$$

$$(2p)_{\text{máx}} = \cancel{2}\cdot 3\cdot\frac{\sqrt{3}}{\cancel{2}} = 3\sqrt{3}$$

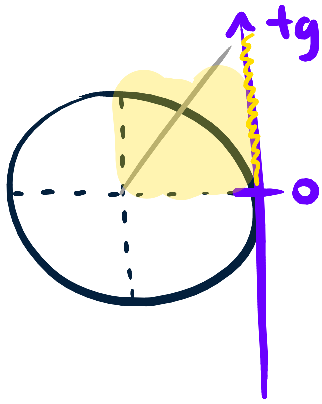
EXEMPLO 02 (IMO)

Dados a, b e $c \in \mathbb{R}_+^*$ dado que $a+b+c = abc$,

mostre que $\dfrac{1}{\sqrt{1+a^2}} + \dfrac{1}{\sqrt{1+b^2}} + \dfrac{1}{\sqrt{1+c^2}} \leq \dfrac{3}{2}$

R :

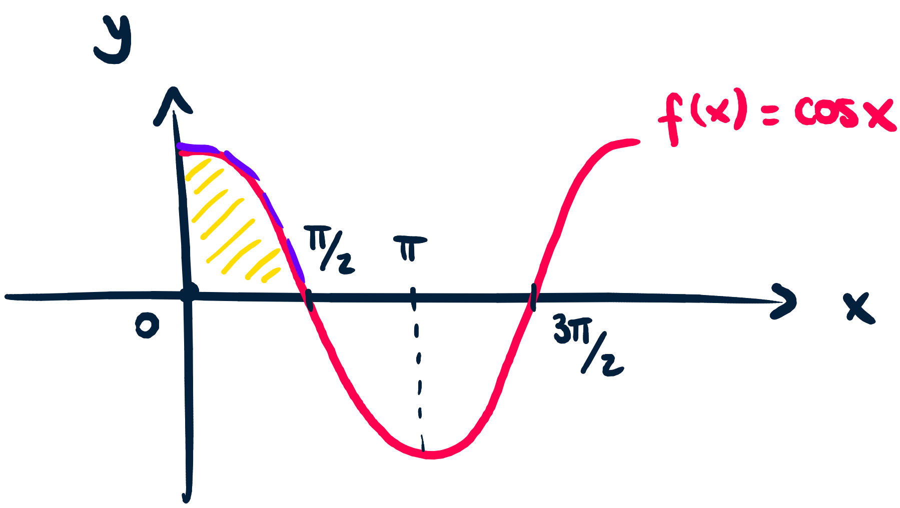
Seja $\left.\begin{aligned} tg\,\alpha &= a \\ tg\,\beta &= b \\ tg\,\gamma &= c \end{aligned}\right\}$ $\alpha, \beta, \gamma \in \left]0, \pi/2\right[$



Logo, usando $1 + tg^2\theta = sec^2\theta$

$$E = \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} =$$

$$= \cos\alpha + \cos\beta + \cos\gamma$$



$$a+b+c = abc \longrightarrow tg\alpha + tg\beta + tg\gamma = tg\alpha \cdot tg\beta \cdot tg\gamma$$

$$\alpha+\beta+\gamma = \pi$$

$f(x) = \cos(x)$

$$\frac{f(\alpha)+f(\beta)+f(\gamma)}{3} \leq f\left(\frac{\alpha+\beta+\gamma}{3}\right)$$

$$\frac{\cos\alpha + \cos\beta + \cos\gamma}{3} \leq \cos\left(\frac{\pi}{3}\right)$$

$$\cos\alpha + \cos\beta + \cos\gamma \leq 3 \cdot \frac{1}{2}$$

$$\boxed{E \leq \frac{3}{2}}$$

$a = b = c = tg\,60^\circ = \sqrt{3}$

SUBSTITUIÇÕES TRIGONOMÉTRICAS

EXEMPLO 01

Encontre o menor valor de

$$x\cdot y + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)}$$

R:

$$\sqrt{1-y^2} \rightarrow \text{C.E.}: 1-y^2 \geq 0 \therefore y^2 \leq 1$$

$$0 \leq y^2 \leq 1$$

$$-1 \leq y \leq 1$$

$$x = \operatorname{sen}\theta$$
$$y = \cos\varphi$$

$$E = x\cdot y + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)}$$

$$= \cancel{\operatorname{sen}\theta\cdot\cos\varphi} + \operatorname{sen}\theta\cdot\operatorname{sen}\varphi + \cos\varphi\cdot\cos\theta - \cancel{\operatorname{sen}\theta\cdot\cos\varphi}$$

$$= \cos(\varphi-\theta) \longrightarrow E_{min} = -1$$

EXEMPLO 02

Mostre que $\dfrac{x}{\sqrt{1+x^2}} + \dfrac{y}{\sqrt{1+y^2}} + \dfrac{z}{\sqrt{1+z^2}} \leq \dfrac{3\sqrt{3}}{2}$

se $x + y + z = x \cdot y \cdot z$

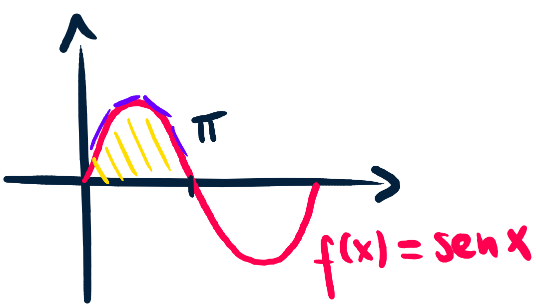
R: $E = \dfrac{x}{\sqrt{1+x^2}} + \dfrac{y}{\sqrt{1+y^2}} + \dfrac{z}{\sqrt{1+z^2}}$

Seja $x = tg\alpha$, $y = tg\beta$, $z = tg\gamma$, logo:

$$x+y+z = x \cdot y \cdot z \rightsquigarrow \underbrace{tg\alpha + tg\beta + tg\gamma = tg\alpha \cdot tg\beta \cdot tg\gamma}_{\alpha + \beta + \gamma = \pi}$$

Logo: $E = \dfrac{tg\alpha}{sec\alpha} + \dfrac{tg\beta}{sec\beta} + \dfrac{tg\gamma}{sec\gamma}$

$$= sen\alpha + sen\beta + sen\gamma \leq 3\sqrt{3}/2$$



$$E \leq 3\sqrt{3}/2$$

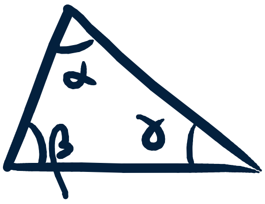
DESIGUALDADE DAS MÉDIAS

$$M.A. \geq M.G. \geq M.H.$$

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}}$$

EXEMPLO

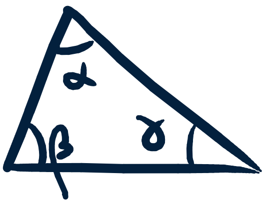
Mostre que $tg\alpha \cdot tg\beta \cdot tg\gamma \geq 3\sqrt{3}$



R: M.A. $\geq$ M.G.

$$\frac{tg\alpha + tg\beta + tg\gamma}{3} \geq \sqrt[3]{tg\alpha \cdot tg\beta \cdot tg\gamma}$$

$$tg\alpha + tg\beta + tg\gamma \geq 3 \cdot \sqrt[3]{tg\alpha \cdot tg\beta \cdot tg\gamma}$$



$$tg\alpha + tg\beta + tg\gamma = tg\alpha \cdot tg\beta \cdot tg\gamma$$

$$(tg\alpha \cdot tg\beta \cdot tg\gamma) \geq 3 \cdot \sqrt[3]{tg\alpha \cdot tg\beta \cdot tg\gamma}$$

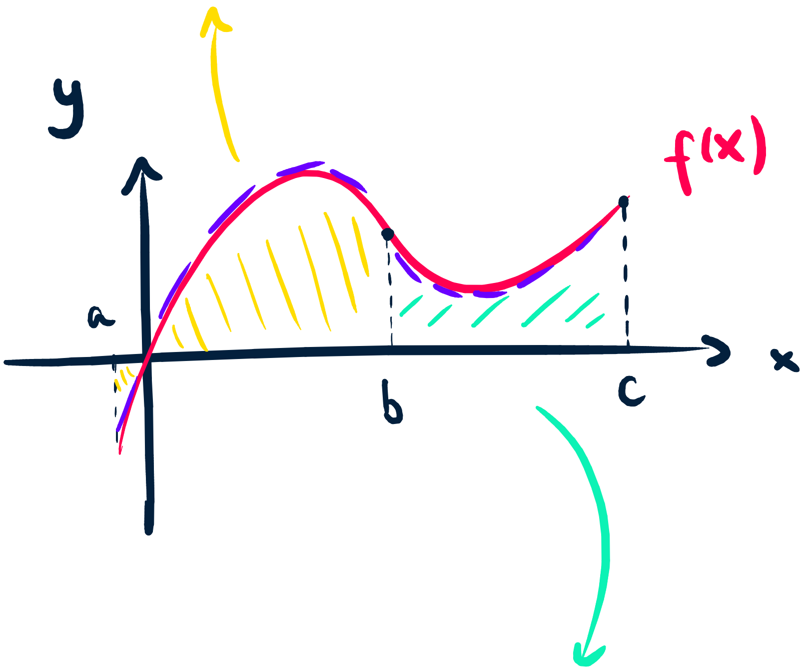
$$(tg\alpha \cdot tg\beta \cdot tg\gamma)^{\cancel{3}\,2} \geq 3^3 \cdot (\cancel{tg\alpha \cdot tg\beta \cdot tg\gamma})$$

$$(tg\alpha \cdot tg\beta \cdot tg\gamma)^2 \geq 3 \cdot 9 \quad \therefore \quad tg\alpha \cdot tg\beta \cdot tg\gamma \geq 3\sqrt{3}$$

RESUMO

CÔNCAVA : $x \in [a,b]$

$$\boxed{\frac{f(a_1)+f(a_2)+\dots+f(a_n)}{n} \leq f\left(\frac{a_1+a_2+\dots+a_n}{n}\right)}$$



CONVEXA : $x \in [b,c]$

$$\boxed{\frac{f(a_1)+f(a_2)+\dots+f(a_n)}{n} \geq f\left(\frac{a_1+a_2+\dots+a_n}{n}\right)}$$