Exercício 1. Escreva uma expressão para a soma de Riemann das funções abaixo no intervalo dado (não é necessário calcular o limite):

(a) $f(x) = \dfrac{2x}{x^2+1}$, $1 \leqslant x \leqslant 3$ (b) $f(x) = x^2 + \sqrt{1+2x}$, $4 \leqslant x \leqslant 7$

(c) $f(x) = \sqrt{\operatorname{sen} x}$, $0 \leqslant x \leqslant \pi$

(a) Vamos dividir $[1,3]$ em n sub intervalos de comprimento $\Delta x = \frac{3-1}{n} = \frac{2}{n}$ usando a partição

$$x_k = 1 + k \cdot \Delta x = 1 + \frac{2k}{n} \qquad (0 \leq k \leq n)$$

$$x_0 = 1 < x_1 = 1 + \frac{2}{n} < x_2 = 1 + \frac{4}{n} < \cdots < x_n = 3$$

Escolhendo $x^*_k \in [x_{k-1}, x_k]$ como

$$x^*_k = x_k = 1 + \frac{2k}{n} \quad (0 < k \leq n),$$

formamos a seguinte soma de Riemann:

$$\sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n} \frac{2x_k}{x_k^2+1}\,\Delta x$$

$$= \sum_{k=1}^{n} \frac{2\left(1+\frac{2k}{n}\right)}{\left(1+\frac{2k}{n}\right)^2+1} \cdot \frac{2k}{n}$$

(b) $\Delta x = \dfrac{7-4}{n} = \dfrac{3}{n}$

$$x_k = 4 + \frac{3k}{n} = x_k^*$$

A soma de Riemann é

$$\sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left[\left(4+\frac{3k}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3k}{n}\right)}\right]\cdot\frac{3k}{n}$$

(c) $\Delta x = \dfrac{\pi}{n}$, $x_k = \dfrac{k\pi}{n} = x_k^*$

Soma de Riemann:

$$\sum_{k=1}^{n} \sqrt{\operatorname{sen}\left(\frac{k\pi}{n}\right)}\cdot\frac{k\pi}{n}$$

Exercício 2. Escreva as expressões abaixo na forma de integral $\int_a^b f(x)dx$, determinando a, b e $f(x)$:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$ (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$

(a) Repare que se

$a = 1$, $b - a = 3 \Rightarrow b = 4$

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$x_i = a + i \cdot \Delta x = 1 + \frac{3i}{n}$

a soma é

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\,\Delta x = \int_1^4 \sqrt{x}\,dx$$

(b) Aqui temos

$a = 0$, $b = \pi/4$

$\Delta x = \frac{\pi}{4n}$, $x_i = i\Delta x = \frac{i\pi}{4n}$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} (\tan x_i)\,\Delta x = \int_0^{\pi/4} \tan x\,dx$$

Exercício 3. Expresse como um limite a área sob a curva $y = x^3$ de $x = 0$ a $x = 1$ e calcule o seu valor, usando o fato de que

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Vamos fazer a partição com $\Delta x = \frac{1}{n}$,

$$x_k = k\Delta x = \frac{k}{n}$$

Temos a soma

$$\lim_{n\to\infty} \sum_{k=1}^{n} (x_k^3)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^3} \cdot \frac{1}{n}$$

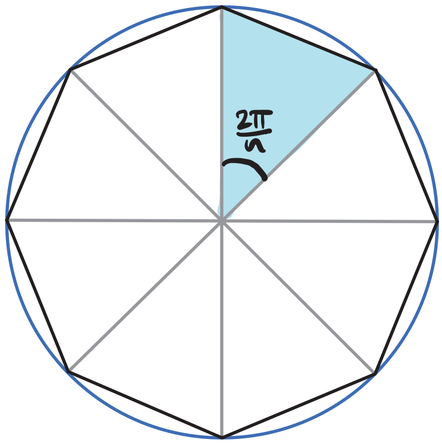
$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \lim_{n\to\infty} \frac{1}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2$$

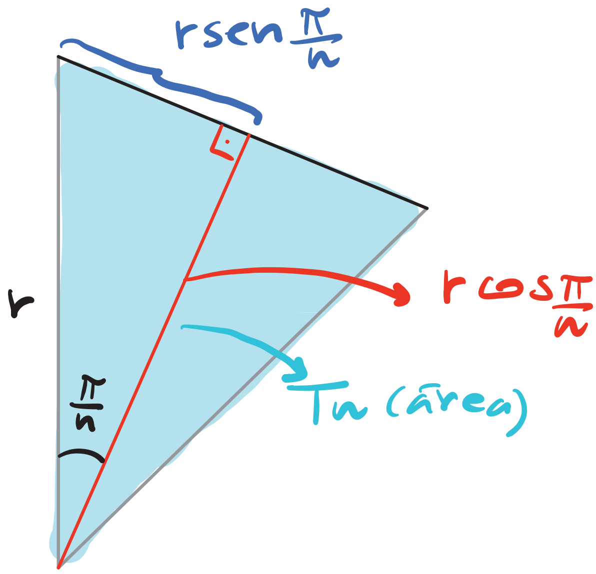
$$= \lim_{n\to\infty} \left[\frac{1}{2}\left(1 + \frac{1}{n}\right)\right]^2 = \frac{1}{4}$$

Exercício 4. Seja A_n a área de um polígono regular de n lados inscrito em um círculo de raio r. Dividindo o polígono em n triângulos congruentes com ângulo central de $2\pi/n$, mostre que

$$A_n = \frac{1}{2} n r^2 \operatorname{sen}\left(\frac{2\pi}{n}\right)$$

e calcule $\lim_{n\to\infty} A_n$.





A área do triângulo é

$$T_n = \frac{1}{2}\, 2r\operatorname{sen}\frac{\pi}{n} \cdot r\cos\frac{\pi}{n}$$

$$= \frac{r^2}{2}\operatorname{sen}\left(\frac{2\pi}{n}\right)$$

Logo,

$$A_n = n \cdot T_n = \frac{1}{2} n r^2 \operatorname{sen}\left(\frac{2\pi}{n}\right)$$

Temos que

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \pi r^2 \cdot \frac{\operatorname{sen}\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} = \pi r^2,$$

pois $\lim\limits_{x \to 0} \dfrac{\text{sen}\, x}{x} = 1$ e $\lim\limits_{n \to \infty} \dfrac{2\pi}{n} = 0$.

Exercício 5. Expresse o limite como uma integral definida no intervalo dado:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1+x_i}\Delta x$, $[0,1]$ (b) $\lim_{n\to\infty} \sum_{i=1}^{n} x_i\sqrt{1+x_i^3}\Delta x$, $[2,5]$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \log(1+x_i^2)\Delta x$, $[2,6]$ (d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\Delta x$, $[\pi,2\pi]$

(a) $\int_0^1 \frac{e^x}{1+x}\,dx$ (b) $\int_2^5 x\sqrt{1+x^3}\,dx$

(c) $\int_2^6 x\log(1+x)\,dx$ (d) $\int_\pi^{2\pi} \frac{\cos x}{x}\,dx$

Exercício 6. Usando o limite de somas de Riemann, calcule as seguintes integrais:

(a) $\displaystyle\int_{-1}^{5}(1+3x)\,dx$ (b) $\displaystyle\int_{-2}^{0}(x^2+x)\,dx$ (c) $\displaystyle\int_{0}^{2}(2x-x^3)\,dx$

(a) $\Delta x = \dfrac{5-(-1)}{n} = \dfrac{6}{n}$

$$x_k = -1 + k\Delta x = -1 + \frac{6k}{n}$$

$$\int_{-1}^{5}(1+3x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}(1+3x_k)\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1-3+\frac{18k}{n}\right)\cdot\frac{6}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} -\frac{12}{n} + \frac{108k}{n^2}$$

$$= -12 + \lim_{n\to\infty}\frac{108}{n^2}\sum_{k=1}^{n}k$$

$$= -12 + \lim_{n\to\infty}\frac{108}{n^2}\,\frac{n(n+1)}{2}$$

$$= -12 + \lim_{n\to\infty} 54\cdot\left(1+\frac{1}{n}\right) = -12+54 = 42.$$

(b) $\Delta x = \dfrac{2}{n}$, $x_k = -2 + \dfrac{2k}{n}$

$$\int_{-2}^{0}(x^2+x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}(x_k^2+x_k)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(-2+\frac{2k}{n}\right)^2 + \left(-2+\frac{2k}{n}\right)\right]\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[4 - \frac{8k}{n} + \frac{4k^2}{n^2} - 2 + \frac{2k}{n}\right]\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(4 - \frac{6k}{n} + \frac{4}{n^2}k^2\right)$$

$$= \lim_{n\to\infty} 8 - \frac{12}{n^2}\sum_{k=1}^{n} k + \frac{8}{n^3}\sum_{k=1}^{n} k^2$$

Lembrando que:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad e \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 8 + \lim_{n\to\infty} -\frac{12}{n^2}\,\frac{n(n+1)}{2} + \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= 8 + \lim_{n\to\infty} -6\left(1+\frac{1}{n}\right) + \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= 8 - 6 + \frac{4}{3}\cdot 2 = 2 + \frac{8}{3} = \frac{14}{3}$$

(c) $\Delta x = \frac{2}{n}$, $x_k = \frac{2k}{n}$

$$\int_0^2 2x - x^3\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[2\left(\frac{2k}{n}\right) - \left(\frac{2k}{n}\right)^3 \right] \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{8}{n^2} k - \frac{16}{n^4} k^3 \right]$$

$$= \lim_{n\to\infty} \frac{8}{n^2} \sum_{k=1}^{n} k - \frac{16}{n^4} \sum_{k=1}^{n} k^3$$

$$= \lim_{n\to\infty} \frac{8}{n^2} \cdot \frac{n(n+1)}{2} - \frac{16}{n^4} \cdot \left[\frac{n(n+1)}{2} \right]^2$$

$$= \lim_{n\to\infty} 4\left(1 + \frac{1}{n}\right) - 4\left(1 + \frac{1}{n}\right)^2$$

$$= 4 - 4 = 0$$

Exercício 7. Dado que $\int_0^{\pi} \operatorname{sen}^4 x\, dx = 3\pi/8$, calcule $\int_{\pi}^{0} \operatorname{sen}^4 \theta\, d\theta$.

Lembre que a letra não importa

e que $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$. Logo,

$$\int_{\pi}^{0} \operatorname{sen}^4 \theta\, d\theta = -\int_0^{\pi} \operatorname{sen}^4 \theta\, d\theta = -\frac{3\pi}{8}$$

Exercício 8. Usando as propriedades básicas das integrais, demonstre (sem calcular as integrais) as seguintes desigualdades:

(a) $\int_0^4 x^2 - 4x + 4\, dx \geqslant 0$ (b) $\int_0^1 \sqrt{1+x^2} \leqslant \int_0^1 \sqrt{1+x}\, dx$

(c) $2 \leqslant \int_{-1}^1 \sqrt{1+x^2}\, dx \leqslant 2\sqrt{2}$ (d) $\frac{\pi}{12} \leqslant \int_{\pi/6}^{\pi/3} \operatorname{sen} x\, dx \leqslant \frac{\sqrt{3}\pi}{12}$

(a) $x^2 - 4x + 4 = (x-2)^2 \geqslant 0 \quad \forall\, x \in \mathbb{R}.$

Logo,

$$\int_0^4 x^2 - 4x + 4\, dx \geqslant \int_0^4 0\, dx = 0 \cdot \int_0^4 dx = 0.$$

(b) Se $0 \leq x \leq 1$,

$$x^2 \leq x \Rightarrow 1 + x^2 \leq 1 + x \Rightarrow \sqrt{1+x^2} \leq \sqrt{1+x}$$

Daí,

$$\int_0^1 \sqrt{1+x^2}\, dx \leq \int_0^1 \sqrt{1+x}\, dx$$

(c) $\forall\, x \in [-1, 1]$, temos

$$1 \leq 1 + x^2 \leq 2$$

$$\Rightarrow 1 \leq \sqrt{1+x^2} \leq \sqrt{2}$$

$$\Rightarrow 2 = 1 \int_{-1}^1 dx = \int_{-1}^1 1\, dx \leq \int_{-1}^1 \sqrt{1+x^2}\, dx \leq$$

$$\leq \int_{-1}^{1} \sqrt{2}\, dx = \sqrt{2} \int_{-1}^{1} dx = 2\sqrt{2}$$

(d) Temos

$$\frac{1}{2} \leq \operatorname{sen} x \leq \frac{\sqrt{3}}{2} \quad \forall \ x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

Logo,

$$\frac{\pi}{12} = \int_{\pi/6}^{\pi/3} \frac{1}{2}\, dx \leq \int_{\pi/6}^{\pi/3} \operatorname{sen} x\, dx \leq \int_{\pi/6}^{\pi/3} \frac{\sqrt{3}}{2}\, dx = \frac{\sqrt{3}\pi}{12}$$

Exercício 9. Expresse o limite como uma integral definida:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ (b) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$

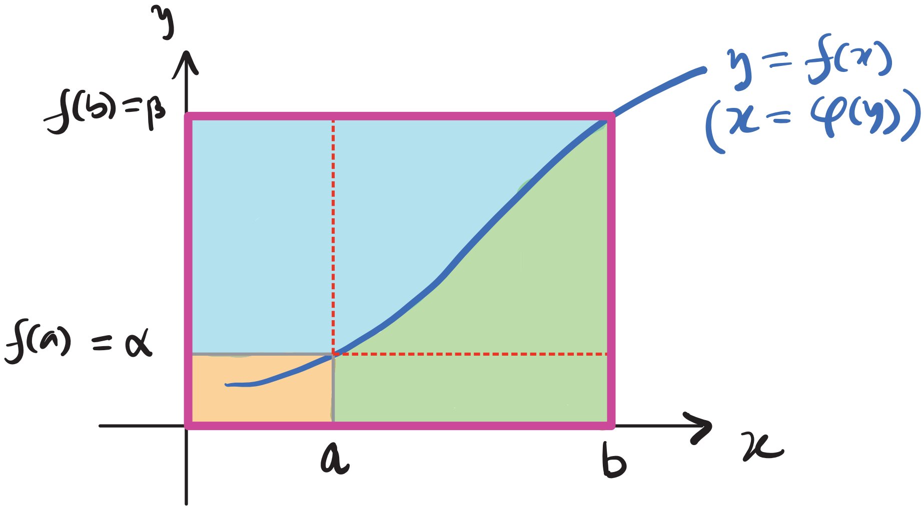
(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 \cdot \frac{1}{n}$

$= \int_0^1 x^4 dx$

(b) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2} dx$

Exercício 10. Seja f positiva e monótona em $[a, b]$, onde $0 < a < b$. Seja φ a inversa de f, $\alpha = f(a)$ e $\beta = f(b)$. Usando a interpretação da integral como uma área, verifique que

$$\int_{\alpha}^{\beta} \varphi(y)dy = b\beta - a\alpha - \int_{a}^{b} f(x)\, dx$$



Vemos que a área azul, que representa $\int_{\alpha}^{\beta} \varphi(y)\, dy$, pode ser obtida a partir do grande retângulo rosa, de área $b \cdot \beta$, subtraindo o retângulo laranja, de área $a\alpha$, e também a área representada pela integral $\int_{a}^{b} f(x)\, dx$. Ou seja:

$$\int_{\alpha}^{\beta} \varphi(y)\, dy = b\beta - a\alpha - \int_{a}^{b} f(x)\, dx$$

Exercício 11. Use o exercício anterior para calcular $(1 < a < b)$

$$\int_a^b \log x \, dx$$

A invers de $\log x$ é e^x. Daí,

$$\int_a^b \log x \, dx = b \log b - a \log a - \int_{\log a}^{\log b} e^x \, dx$$

$$= b \log b - a \log a - \left[e^{\log b} - e^{\log a}\right]$$

$$= b \log b - a \log a - b + a$$

Exercício 12. O que podemos dizer sobre $\int_{-a}^{a} f(x)\,dx$ quando f é ímpar? E quando f é par?

Se $f(x)$ é ímpar, então

$$f(-x) = -f(x)$$

Portanto,

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Se f é par, $f(-x) = f(x)$, e daí

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

Exercício 13. Avalie

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}\right)$$

Temos

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{k}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{\frac{k}{n}}} \cdot \frac{1}{n}$$

$$= \int_0^1 x^{-1/2}\,dx = \frac{1^{1/2} - 0^{1/2}}{1/2} = 2$$

Exercício 14. (*Desigualdade de Cauchy-Schwarz*) Mostre que, para qualquer sequência de números reais $a_1, ..., a_n$ e $b_1, ..., b_n$,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqslant \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

calculando o mínimo da função

$$g(t) = \sum_{i=1}^{n}(a_i + tb_i)^2 \geqslant 0$$

Temos

$$g'(t) = \sum_{i=1}^{n} 2b_i(a_i + tb_i)$$

$$= 2t\sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i = 0$$

se

$$t = \frac{-\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i^2} = t^*$$

Como g é um polinômio quadrático com coeficiente em t^2 igual a $\sum_{i=1}^{n} b_i^2 > 0$, segue que seu ponto crítico é um mínimo global.

Seja $\alpha = \sum_{i=1}^{n} a_i^2$, $\beta = \sum_{i=1}^{n} b_i^2$ e $\gamma = \sum_{i=1}^{n} a_i b_i$. Temos então que

$$g(t) = \sum_{i=1}^{n} (a_i^2 + 2ta_ib_i + b_i^2)$$

$$= \alpha + 2t\gamma + \beta t^2 \geq 0$$

Logo,

$$g(t^*) = \alpha + 2\gamma \cdot \left(-\frac{\gamma}{\beta}\right) + \beta\left(-\frac{\gamma}{\beta}\right)^2$$

$$= \alpha - \frac{2\gamma^2}{\beta} + \frac{\gamma^2}{\beta}$$

$$= \frac{\alpha\beta - \gamma^2}{\beta} \geq 0$$

Como $\beta = \sum_{i=1}^{n} b_i^2 > 0$, segue que

$$\alpha\beta - \gamma^2 \geq 0,$$

ou seja,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Exercício 15. (*Desigualdade de Cauchy-Schwarz para integrais*) Mostre que, se $f(x), g(x)$ são funções contínuas, então

$$\int_a^b [f(x)]^2\,dx \int_a^b [g(x)]^2\,dx \geqslant \left(\int_a^b f(x)g(x)\,dx\right)^2$$

Escrevendo somas de Riemann para as integrais, temos, para $\Delta x = \frac{b-a}{n}$,

$$\int_a^b [f(x)]^2 dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^n f(x_i)^2$$

$$\int_a^b [g(x)]^2 dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^n g(x_i)^2$$

$$\int_a^b f(x)\,g(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^n f(x_i)\,g(x_i)$$

Por Cauchy-Schwarz (exercício anterior),

$$\left(\sum_{i=1}^n f(x_i)^2\right)\left(\sum_{i=1}^n g(x_i)^2\right) \geqslant \left(\sum_{i=1}^n f(x_i)\,g(x_i)\right)^2$$

$$\Rightarrow \left(\Delta x\sum_{i=1}^n f(x_i)^2\right)\left(\Delta x\sum_{i=1}^n g(x_i)^2\right) \geqslant \left(\Delta x\sum_{i=1}^n f(x_i)\,g(x_i)\right)^2$$

$$\Rightarrow \lim_{n\to\infty} \left(\Delta x\sum_{i=1}^n f(x_i)^2\right)\left(\Delta x\sum_{i=1}^n g(x_i)^2\right) \geqslant \lim_{n\to\infty} \left(\Delta x\sum_{i=1}^n f(x_i)\,g(x_i)\right)^2$$

$$\Rightarrow \left(\lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)^2\right)\left(\lim_{n\to\infty} \Delta x \sum_{i=1}^{n} g(x_i)^2\right) \geqslant \left(\lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)\, g(x_i)\right)^2$$

$$\Rightarrow \left(\int_a^b f(x)^2\, dx\right)\left(\int_a^b g(x)^2\, dx\right) \geqslant \left(\int_a^b f(x)\, g(x)\, dx\right)^2$$

Exercício 16. Seja f contínua e positiva em $[a, b]$ e seja M o seu valor máximo. Prove que

$$M = \lim_{n\to\infty} \sqrt[n]{\int_a^b [f(x)]^n \, dx}$$

Primeiramente,

$$\sqrt[n]{\int_a^b f(x)^n dx} \leq \sqrt[n]{\int_a^b M^n dx} = M \cdot \sqrt[n]{(b-a)} \xrightarrow[n\to\infty]{} M$$

Logo, $\lim\limits_{n\to\infty} \sqrt[n]{\int_a^b f(x)^n dx} \leq M.$

Seja $m \in [a,b]$ tal que $f(m) = M$ (existe pelo Teorema dos Extremos de Weierstrass).

Então, como f é contínua, dado $\varepsilon > 0$, $\exists\, \delta > 0$ tal que se $x \in (m - \delta/2, m + \delta/2)$ então $1 - \varepsilon < \frac{f(x)}{M} < 1 + \varepsilon$. Logo,

$$\sqrt[n]{\int_a^b f(x)^n dx} = M \sqrt[n]{\int_a^b \left(\frac{f(x)}{M}\right)^n dx}$$

$$\geq M \sqrt[n]{\int_{m-\delta/2}^{m+\delta/2} (1-\varepsilon)^n dx}$$

$$= M(1-\varepsilon)\sqrt[n]{\delta} \xrightarrow[n\to\infty]{} M(1-\varepsilon).$$

Daí,

$$M \geq \lim_{n\to\infty} \sqrt[n]{\int_a^b f(x)^n\,dx} \geq M(1-\varepsilon)$$

Como $\varepsilon > 0$ é arbitrário, segue que

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b f(x)^n\,dx} = M.$$

pelo teorema do sanduíche.

Exercício 17. Calcule $\lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$.

Temos

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \frac{1}{n}$$

$$= \int_1^2 \frac{1}{x}\,dx = \log 2 - \log 1 = \log 2.$$

Exercício 1. Calcule $g'(x)$ de duas formas: usando o TFC parte 1 e calculando a integral usando o TFC parte 2 e derivando.

(a) $g(x) = \int_1^x t^2 dt$ (b) $g(x) = \int_0^x 2 + \operatorname{sen} t dt$

(a) TFC 1: $g'(x) = x^2$

TFC 2: $g(x) = \int_1^x t^2 dt = \frac{t^3}{3}\Big|_1^x = \frac{x^3}{3} - \frac{1}{3}$

$\Rightarrow g'(x) = x^2$

(b) TFC 1: $g'(x) = 2 + \operatorname{sen} x$

TFC 2: $g(x) = \int_0^x 2 + \operatorname{sen} t \, dt = 2t - \cos t \Big|_0^x$

$= 2x - \cos x + 1$

$\Rightarrow g'(x) = 2 + \operatorname{sen} x$

Exercício 2. Encontre as derivadas das seguintes funções:

(a) $f(x) = \int_0^x \sqrt{1+\sec t}\,dt$ (b) $h(x) = \int_1^{e^x} \log t\,dt$ (c) $y = \int_1^{3x+1} \frac{1}{1+t^3}\,dt$

(d) $y = \int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt$ (e) $y = \int_{\sqrt{x}}^{\pi/4} \theta \tan\theta\,d\theta$ (f) $y = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$

(g) $f(x) = \int_{1-2x}^{1+2x} t \operatorname{sen} t\,dt$ (h) $g(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt$ (i) $\int_{\cos x}^{\operatorname{sen} x} \log(1+2v)\,dv$

(a) $f'(x) = \sqrt{1+\sec x}$

(b) Se $g(u) = \int_1^u \log t\,dt$, temos

$$h(x) = g(e^x)$$

$$\Rightarrow h'(x) = g'(e^x)\cdot \frac{d}{dx}e^x$$

$$= \log(e^x)\cdot e^x$$

(c) $y' = \dfrac{3}{1+(3x+1)^3}$

(d) $y' = \sec^2 x \sqrt{\tan x + \sqrt{\tan x}}$

(e) $y' = -\dfrac{1}{2\sqrt{x}} \sqrt{x} \tan\sqrt{x}$

(f) $y = \int_0^{3x} \dfrac{u^2-1}{u^2+1}\,du - \int_0^{2x} \dfrac{u^2-1}{u^2+1}\,du$

$$\Rightarrow y' = \frac{(3x)^2-1}{(3x)^2+1} - \frac{(2x)^2-1}{(2x)^2+1}$$

(g) $f(x) = \int_0^{1+2x} t \operatorname{sen} t \, dx - \int_0^{1-2x} t \operatorname{sen} t \, dt$

$$\Rightarrow f'(x) = 2\left[(1+2x)\operatorname{sen}(1+2x) + (1-2x)\operatorname{sen}(1-2x)\right]$$

(h) $g(x) = \int_0^{2x} \arctan t \, dt - \int_0^{\sqrt{x}} \arctan t \, dt$

$$\Rightarrow g'(x) = 2 \arctan 2x - \frac{1}{2\sqrt{x}} \arctan \sqrt{x}$$

(i) $y = \int_0^{\operatorname{sen} x} \log(1+2v) \, dv - \int_0^{\cos x} \log(1+2v) \, dv$

$$\Rightarrow y' = \cos x \log(1+2\operatorname{sen} x) + \operatorname{sen} x \log(1+2\cos x)$$

Exercício 3. Calcule a integral

(a) $\int_1^3 x^2 + 2x - 4\,dx$ (b) $\int_{-1}^1 x^{100}\,dx$ (c) $\int_0^1 x^{4/5}\,dx$

(d) $\int_{\pi/6}^{\pi} \operatorname{sen}\theta\,d\theta$ (e) $\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx$ (f) $\int_{\pi/6}^{\pi/4} \operatorname{cossec} t \cot t\,dt$

(g) $\int_1^2 \frac{v^3+3v^6}{v^4}\,dv$ (h) $\int_0^1 x^e + e^x\,dx$ (i) $\int_{-1}^1 e^{u+1}\,du$

(j) $\int_0^4 2^s\,ds$ (k) $\int_1^8 \sqrt[3]{x}\,dx$ (l) $\int_{\pi}^{2\pi} \cos\theta\,d\theta$

(m) $\int_{-1}^2 (3u-2)(u+1)\,du$ (n) $\int_{\pi/4}^{\pi/3} \operatorname{cossec}^2\theta\,d\theta$ (o) $\int_0^3 2\operatorname{sen} x - e^x\,dx$

(p) $\int_0^1 \cosh t\,dt$ (q) $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx$

(a) $\int_1^3 x^2+2x-4\,dx = \frac{x^3}{3} + x^2 - 4x \Big|_1^3$

$= 9+9-12-\frac{1}{3}-1+4 = \frac{26}{3}$

(b) $\int_{-1}^1 x^{100}\,dx = \frac{1}{101}x^{101}\Big|_{-1}^1 = \frac{2}{101}$

(c) $\int_0^1 x^{4/5}\,dx = \frac{5}{9}x^{9/5}\Big|_0^1 = \frac{5}{9}$

(d) $\int_{\pi/6}^{\pi} \operatorname{sen}\theta\,d\theta = -\cos\theta\Big|_{\pi/6}^{\pi} = 1 + \frac{\sqrt{3}}{2}$

(e) $\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx = \int_1^4 2x^{-1/2} + x^{3/2}\,dx$

$= 4x^{1/2} + \frac{2}{5}x^{5/2}\Big|_1^4 = 8 + \frac{64}{5} - 4 - \frac{2}{5}$

$= \frac{82}{5}$

(f) $\displaystyle\int_{\pi/6}^{\pi/4} \operatorname{cossec} t \cot t \, dt = -\operatorname{cossec} t \Big|_{\pi/6}^{\pi/4}$

$= 2 - \sqrt{2}$

(g) $\displaystyle\int_1^2 \frac{v^3 + 3v^6}{v^4} dv = \int_1^2 \frac{1}{v} + 3v^2 dv$

$= \log v + v^3 \Big|_1^2 = \log 2 + 7$

(h) $\displaystyle\int_0^1 x^e + e^x dx = \frac{x^{e+1}}{e+1} + e^x \Big|_0^1$

$\displaystyle = \frac{1}{e+1} + e - 1 = \frac{\cancel{1} + e^2 + \cancel{e} - \cancel{e} - \cancel{1}}{e+1}$

$\displaystyle = \frac{e^2}{e+1}$

(i) $\displaystyle\int_{-1}^{1} e^{u+1} du = e\int_{-1}^{1} e^u du = e \cdot e^u \Big|_{-1}^{1} = e^2 - 1$

(j) $\displaystyle\int_0^4 2^s ds = \frac{1}{\log 2} 2^s \Big|_0^4 = \frac{15}{\log 2}$

(k) $\displaystyle\int_1^8 x^{1/3} dx = \frac{3}{4} x^{4/3} \Big|_1^8 = 12 - \frac{3}{4} = \frac{45}{4}$

(ℓ) $\displaystyle\int_{\pi}^{2\pi} \cos\theta \, d\theta = \operatorname{sen}\theta \Big|_{\pi}^{2\pi} = 0$

(m) $\displaystyle\int_{-1}^{2}(3u-2)(u+1)\,du = \int_{-1}^{2} 3u^2+u-2\,du$

$= u^3+\frac{1}{2}u^2-2u\Big|_{-1}^{2} = 8+\cancel{2}-4+1-\frac{1}{2}-\cancel{2}$

$= \frac{9}{2}$

(n) $\displaystyle\int_{\pi/4}^{\pi/3}\operatorname{cossec}^2\theta\,d\theta = -\cot\theta\Big|_{\pi/4}^{\pi/3} = 1-\frac{1}{\sqrt{3}}$

(o) $\displaystyle\int_{0}^{3} 2\operatorname{sen}x - e^x\,dx = -2\cos x - e^x\Big|_{0}^{3}$

$= 3 - 2\cos(3) - e^3$

(p) $\displaystyle\int_{0}^{1}\cosh t\,dt = \operatorname{senh} t\Big|_{0}^{1} = \frac{e^2-1}{2e}$

(q) $\displaystyle\int_{1/2}^{1/\sqrt{2}}\frac{4}{\sqrt{1-x^2}}\,dx = 4\cdot\operatorname{arcsen} x\Big|_{1/2}^{1/\sqrt{2}}$

$= 4\cdot\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{4\pi}{12} = \frac{\pi}{3}$

Exercício 4. Em que intervalo a curva

$$y = \int_0^x \frac{t^2}{t^2 + t + 2}\, dt$$

é côncava para baixo?

$$y' = \frac{x^2}{x^2 + x + 2}$$

$$y'' = \frac{2x(x^2 + x + 2) - x^2(2x + 1)}{(x^2 + x + 2)^2}$$

$$= \frac{\cancel{2x^3} + 2x^2 + 4x - \cancel{2x^3} - x^2}{(x^2 + x + 2)^2}$$

$$= \frac{x(x+4)}{(x^2+x+2)^2} < 0 \text{ se } x(x+4) < 0$$

Logo, a curva é côncava se

$$-4 < x < 0$$

Exercício 5. Seja $f(x) = \int_2^x e^{t^2} dt$. Encontre uma equação da reta tangente à curva $y = f(x)$ no ponto com coordenada $x = 2$.

A reta tangente tem equação (em $x=2$)

$$y = f'(2)(x-2) + f(2)$$

Temos

$$f'(x) = e^{x^2}$$

$$\Rightarrow f'(2) = e^4$$

e

$$f(2) = \int_2^2 e^{t^2} dt = 0$$

Logo, a reta tangente é

$$y = e^4(x-2)$$

Exercício 6. Calcule os seguintes limites:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2} \right)$

(b) $\lim_{n\to\infty} \frac{1}{n} \left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{n}{n}} \right)$

(c) $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \frac{1}{\sqrt{n}\sqrt{n+3}} + \dots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right)$

(d) $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2-0}} + \frac{1}{n^2-1} + \frac{1}{n^2-4} + \dots + \frac{1}{n^2-(n-1)^2} \right)$

(e) $\lim_{n\to\infty} \frac{1^\alpha + 2^\alpha + 3^\alpha + \dots + n^\alpha}{n^{\alpha+1}}$, onde $\alpha > -1$

(a) $$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2} \right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^4 + \left(\frac{i}{n}\right) \right] \cdot \frac{1}{n}$$

Este é o limite de uma soma de Riemann para a integral

$$\int_0^1 x^4 + x \, dx = \frac{x^5}{5} + \frac{x^2}{2} \Big|_0^1 = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$$

(b) Esse é o limite de uma soma de Riemann para

$$\int_0^1 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

(c) Note que $\frac{1}{\sqrt{n}\sqrt{n+k}} = \frac{1}{n} \cdot \frac{1}{\sqrt{1+\frac{k}{n}}}$

Assim, temos o limite de uma soma de Riemann para a integral

$$\int_1^2 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^2 = 2(\sqrt{2}-1)$$

(d) Note que, para $0 \leq k \leq n-1$,

$$\frac{1}{\sqrt{n^2-k^2}} = \frac{1}{n}\cdot\frac{1}{\sqrt{1-\left(\frac{k}{n}\right)^2}}$$

Assim, temos o limite de uma soma de Riemann para a integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \operatorname{arcsen} 1$$

(e) Temos $\dfrac{k^\alpha}{n^{\alpha+1}} = \dfrac{1}{n}\left(\dfrac{k}{n}\right)^\alpha$

Logo, a soma é uma soma de Riemann para a integral

$$\int_0^1 x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_0^1 = \frac{1}{\alpha+1}$$

Exercício 7. Se f é contínua e g, h são deriváveis, encontre uma fórmula para

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$

Seja a um ponto no domínio de f. Seja

$$F(u) = \int_a^u f(t)\,dt.$$

Então

$$\int_{g(x)}^{h(x)} f(t)\,dt = (F \circ h)(x) - (F \circ g)(x)$$

Daí, pela regra da cadeia,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = F'(h(x))\,h'(x) - F'(g(x))\,g'(x)$$

$$= f(h(x))\,h'(x) - f(g(x))\,g'(x),$$

onde na última linha usamos o Teorema Fundamental do Cálculo.

Exercício 8. Calcule:

(a) $\displaystyle\int x^{1,3} + 7x^{2,5}\,dx$ (b) $\displaystyle\int v(v^2+v)^2\,dv$ (c) $\displaystyle\int \sqrt[4]{x^5}\,dx$

(d) $\displaystyle\int \sqrt{t}(t^2+3t+2)\,dt$ (e) $\displaystyle\int \operatorname{cossec}^2 t - 2e^t\,dt$ (f) $\displaystyle\int 2+\tan^2\theta\,d\theta$

(g) $\displaystyle\int \left(\frac{1+t}{t}\right)^2 dt$ (h) $\displaystyle\int \sec t(\sec t+\tan t)\,dt$ (i) $\displaystyle\int \frac{\operatorname{sen} 2x}{\operatorname{sen} x}\,dx$

(a) $$\int x^{1,3} + 7x^{2,5}\,dx = \frac{x^{2,3}}{2,3} + \frac{7x^{3,5}}{3,5}$$

(b) $$\int v(v^2+v)^2\,dv = \int v^5 + 2v^4 + v^3\,dv$$
$$= \frac{v^6}{6} + \frac{2v^5}{5} + \frac{v^4}{4}$$

(c) $$\int x^{5/4}\,dx = \frac{4}{9}x^{9/4}$$

(d) $$\int t^{1/2}(t^2+3t+2)\,dt = \int t^{5/2} + 3t^{3/2} + 2t^{1/2}\,dt$$
$$= \frac{2}{7}t^{7/2} + \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2}$$

(e) $$\int \operatorname{cossec}^2 t - 2e^t\,dt = -\cot t - 2e^t$$

(f) $$\int 2 + \tan^2\theta\,d\theta = \int 1 + (1+\tan^2\theta)\,d\theta$$
$$= \int 1 + \sec^2\theta\,d\theta = \theta + \tan\theta$$

(g) $$\int \left(\frac{1+t}{t}\right)^2 dt = \int (1+t^{-1})^2\,dt = \int 1 + 2t^{-1} + t^{-2}\,dt$$

$$= t + 2\log t - \frac{1}{t}$$

(h) $\int \sec t\,(\sec t + \tan t)\,dt$

$$= \int \sec^2 t + \sec t \cdot \tan t\,dt$$

$$= \tan t + \sec t$$

(i) $\int \frac{\operatorname{sen} 2x}{\operatorname{sen} x}\,dx = \int \frac{2\cancel{\operatorname{sen} x}\cos x}{\cancel{\operatorname{sen} x}}\,dx$

$$= 2\int \cos x\,dx = 2\operatorname{sen} x$$

Exercício 9. Encontre:

(a) $\displaystyle\int_0^{\pi} 5e^x + 3\,\text{sen}\,x\,dx$ (b) $\displaystyle\int_1^4 \frac{4+6u}{\sqrt{u}}\,du$ (c) $\displaystyle\int_0^1 x(\sqrt[3]{x}+\sqrt[4]{x})\,dx$

(d) $\displaystyle\int_0^1 x^{10} + 10^x\,dx$ (e) $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$ (f) $\displaystyle\int_0^{\pi/3} \frac{\text{sen}\,\theta + \text{sen}\,\theta\tan^2\theta}{\sec^2\theta}\,d\theta$

(g) $\displaystyle\int_0^8 \frac{2+t}{\sqrt[3]{t^2}}\,dt$ (h) $\displaystyle\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}}$ (i) $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$

(j) $\displaystyle\int_{-10}^{10} \frac{2e^x}{\text{senh}\,x + \cosh x}\,dx$ (k) $\displaystyle\int_0^{3\pi/2} |\,\text{sen}\,x|\,dx$

(a) $\displaystyle\int_0^{\pi} 5e^x + 3\,\text{sen}\,x\,dx = 5e^x - 3\cos x\Big|_0^{\pi}$

$= 5e^{\pi} - 5 + 6 = 5e^{\pi} + 1$

(b) $\displaystyle\int_1^4 4u^{-1/2} + 6u^{1/2}\,du = 8u^{1/2} + 4u^{3/2}\Big|_1^4$

$= 16 + 32 - 8 - 4 = 36$

(c) $\displaystyle\int_0^1 x^{4/3} + x^{5/4}\,dx = \frac{3}{7}x^{7/3} + \frac{4}{9}x^{9/4}\Big|_0^1$

$= \dfrac{55}{63}$

(d) $\displaystyle\int_0^1 x^{10} + 10^x\,dx = \frac{x^{11}}{11} + \frac{10^x}{\log 10}\Big|_0^1$

$= \dfrac{1}{11} + \dfrac{10}{\log 10} - \dfrac{1}{\log 10} = \dfrac{1}{11} + \dfrac{9}{\log 10}$

(e) $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} \sec^2\theta + 1\,d\theta$

$= \tan\theta + \theta\Big|_0^{\pi/4} = 1 + \pi/4$

(f) $\displaystyle\int_0^{\pi/3} \frac{\operatorname{sen}\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \frac{\operatorname{sen}\theta \cdot \cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$

$\displaystyle = \int_0^{\pi/3} \operatorname{sen}\theta\,d\theta = -\cos\theta \Big|_0^{\pi/3} = 1 - \frac{1}{2} = \frac{1}{2}$

(g) $\displaystyle\int_0^{8} 2t^{-2/3} + t^{1/3}\,dt = 6t^{1/3} + \frac{3}{4}t^{4/3}\Big|_0^{8}$

$= 12 + 12 = 24$

(h) $\displaystyle\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}} = \operatorname{arcsen} r \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$

(i) $\displaystyle\int_0^{\pi/4} \sec\theta \tan\theta\,d\theta = \sec\theta\Big|_0^{\pi/4} = \sqrt{2} - 1$

(j) Note que

$\displaystyle \operatorname{senh}x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x$

Logo,

$\displaystyle\int_{-10}^{10} \frac{2e^x}{\operatorname{senh}x + \cosh x}\,dx = \int_{-10}^{10} \frac{2e^x}{e^x}\,dx = 2x\Big|_{-10}^{10} = 40$

(k) $\displaystyle\int_0^{3\pi/2} |\operatorname{sen}x|\,dx = \int_0^{\pi} \operatorname{sen}x\,dx - \int_{\pi}^{\frac{3\pi}{2}} \operatorname{sen}x\,dx$

$\displaystyle = -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{3\pi/2} = 1 + 1 + 1 = 3$

Exercício 10. A água escoa pelo fundo de um tanque de armazenamento a uma taxa de $r(t) = 200 - 4t$ litros por minuto, onde $0 \leqslant t \leqslant 50$. Encontre a quantidade de água que escoa do tanque durante os primeiros dez minutos.

A quantidade de água que escoou é dada pela integral

$$Q = \int_0^{10} r(t)\,dt = \int_0^{10} 200 - 4t\,dt = 200t - 2t^2 \Big|_0^{10}$$

$$= 2000 - 200 = 1800 \text{ litros}$$

Exercício 11. O custo marginal de fabricação de x metros de um certo tecido é

$$C'(x) = 3 - 0,01x + 0,000006x^2$$

em dólares por metro. Ache o aumento do custo se o nível de produção for elevado de 2000 para 4000 metros.

$$C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$$

$$= \int_{2000}^{4000} 3 - 0,01x + 0,000006x^2\,dx$$

$$= 3x - \frac{0,01}{2}\cdot x^2 + \frac{0,000006}{3}\cdot x^3 \Big|_{2000}^{4000}$$

$$= 6000 - \frac{10^{-2}}{2}\cdot 10^6 \cdot 12 + \frac{6}{3}\cdot 10^{-6}\cdot 10^9 \cdot 56$$

$$= 6000 - 60.000 + 112.000$$

$$= 58.000$$

Exercício 12. Calcule

$$\lim_{x\to 3} \frac{x}{x-3}\int_3^x \frac{\operatorname{sen} t}{t}\,dt$$

Temos o quociente

$$\frac{x\int_3^x \frac{\operatorname{sen} t}{t}\,dt}{x-3},$$

que gera uma indeterminação $\frac{0}{0}$ quando $x \to 3$. Por L'Hôpital,

$$\lim_{x\to 3} \frac{x\int_3^x \frac{\operatorname{sen} t}{t}\,dt}{x-3} =$$

$$= \lim_{x\to 3} \frac{\int_3^x \frac{\operatorname{sen} t}{t}\,dt + \cancel{x}\,\frac{\operatorname{sen} x}{\cancel{x}}}{1}$$

$$= \operatorname{sen} 3$$

Exercício 13. Mostre que se f for contínua e

$$f(x) = \int_0^x f(t)\,dt,$$

então $f(x)$ é identicamente nula (ou seja, $f(x) = 0$ para todo $x \in \mathbb{R}$).

Temos que, derivando os dois lados da relação,

$$f'(x) = f(x)$$

Assim,

$$\frac{d}{dx}\left(f(x)e^{-x}\right) = \left(f'(x) - f(x)\right)e^{-x} = 0$$

Logo,

$$f(x)e^{-x} = K, \quad \text{constante.}$$

Daí,

$$K = f(0)e^0 = \int_0^0 f(t)\,dt = 0$$

Como $e^{-x} \neq 0 \ \forall x \in \mathbb{R}$, segue que

$$f(x) = 0 \quad \forall x \in \mathbb{R}.$$

Exercício 14.

(a) Verifique que, se $x > -1$, $\log(x+1) = \int_0^x \frac{du}{u+1}$

(b) Mostre que, para todo $n \in \mathbb{N}$, se $0 < x < 1$, então

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \frac{x^{2n}}{2n} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{x^{2n+1}}{2n+1}$$

(*Sugestão: compare* $1/(u+1)$ *com uma progressão geométrica*)

(a) Derivando, temos

$$\frac{d}{dx}\int_0^x \frac{du}{u+1} = \frac{1}{x+1} = \frac{d}{dx}\log(x+1)$$

Logo,

$$\log(x+1) - \int_0^x \frac{du}{u+1} = \text{constante.}$$

Como

$$\log(0+1) = 0 = \int_0^0 \frac{du}{u+1},$$

segue que

$$\log(x+1) = \int_0^x \frac{du}{u+1}$$

(b) Temos que, se $0 < u < 1$,

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$$

$$= \lim_{N\to\infty} \sum_{K=0}^{N} (-u)^K = \lim_{K\to\infty} \frac{1-(-u)^{N+1}}{1+u}$$

$$= \frac{1}{1+u} + \lim_{N\to\infty} \frac{(-u)^{N+1}}{1+u}.$$

Assim, se $N = 2n-1$

$$\sum_{k=0}^{2n-1} (-u)^k = \frac{1}{1+u} - \frac{u^{2n}}{1+u} < \frac{1}{1+u}$$

Por outro lado, se $N = 2n$,

$$\frac{1}{1+u} < \frac{1}{1+u} + \frac{u^{2n+1}}{1+u} = \sum_{k=0}^{2n} (-u)^k$$

Assim, se $0 < x < 1$,

$$\int_0^x \sum_{k=0}^{2n-1} (-u)^k \, du < \int_0^x \frac{du}{1+u} < \int_0^x \sum_{k=0}^{2n} (-u)^k du$$

$$\Rightarrow \sum_{k=0}^{2n-1} \frac{(-u)^{k+1}}{k+1} \Bigg|_0^x < \log(x+1) < \sum_{k=0}^{2n} \frac{(-u)^{k+1}}{k+1} \Bigg|_0^x$$

Ou seja,

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{2n}}{2n} < \log(x+1) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n+1}}{2n+1}$$

Exercício 15. Prove que se f é contínua no intervalo $[0,1]$ então

$$\lim_{x\to 0} x\int_x^1 \frac{f(z)}{z^2}\,dz = f(0)$$

Seja $\varepsilon > 0$. Como f é contínua em $z=0$, $\exists\, \delta > 0$ tal que se $0<z<\delta$, então

$$|f(z)-f(0)|<\varepsilon$$

$$\Rightarrow \quad f(0)-\varepsilon < f(z) < f(0)+\varepsilon \quad \text{se} \quad 0<z<\delta$$

Podemos supor que $\delta < 1$.

Como f é contínua em $[0,1]$, segue do teorema dos extremos que existe $M>0$ tal que

$$|f(z)|<M \quad \forall\ z\in[0,1].$$

Assim, se $0<x<\delta$,

$$x\int_x^1 \frac{f(z)}{z^2}\,dz = x\left[\int_x^\delta \frac{f(z)}{z^2}\,dz + \int_\delta^1 \frac{f(z)}{z^2}\,dz\right], \quad (\mathrm{I})$$

com

$$\left|x\int_\delta^1 \frac{f(z)}{z^2}\,dz\right| \le x\int_\delta^1 \frac{M}{z^2}\,dz \le x\,\frac{M}{\delta^2} \quad (\mathrm{II})$$

Por outro lado,

$$x\int_x^\delta \frac{f(0)-\varepsilon}{z^2}\,dz < x\int_x^\delta \frac{f(z)}{z^2}\,dz < x\int_x^\delta \frac{f(0)+\varepsilon}{z^2}\,dz$$

Como

$$\int_x^\delta z^{-2}\,dz = -z^{-1}\Big|_x^\delta = \frac{1}{x}-\frac{1}{\delta},$$

segue que, se $0<x<\delta$,

$$\left(1-\frac{x}{\delta}\right)(f(0)-\varepsilon) < x\int_x^\delta \frac{f(z)}{z^2}dz < \left(1-\frac{x}{\delta}\right)(f(0)+\varepsilon)$$

Logo, usando também (I) e (II), obtemos

$$-\frac{xM}{\delta^2}+\left(1-\frac{x}{\delta}\right)(f(0)-\varepsilon) < x\int_x^1 \frac{f(z)}{z^2}dz <$$

$$< \frac{xM}{\delta^2}+\left(1-\frac{x}{\delta}\right)(f(0)+\varepsilon)$$

Note que, quando $x\to 0$, o termo à esquerda tende a $f(0)-\varepsilon$, enquanto o termo à direita tende a $f(0)+\varepsilon$.

Assim, podemos encontrar $\xi>0$ suficientemente pequeno tal que se $0<x<\xi$ então

$$f(0)-2\varepsilon < x\int_x^1 \frac{f(z)}{z^2}dz < f(0)+2\varepsilon$$

Como $\varepsilon>0$ é arbitrário, a conclusão é que

$$\lim_{x\to 0} x\int_x^1 \frac{f(z)}{z^2}dz = f(0).$$

Exercício 16. Prove que se uma partícula percorre uma distância de 1 em tempo 1, começando e terminando em repouso, então em algum momento no intervalo de tempo $[0, 1]$ a partícula estava sujeita a uma aceleração igual a 4 ou mais.

Seja $f(t)$ a posição da partícula no tempo t.

Assuma por absurdo que sua aceleração nunca chega em 4, ou seja,

$$|f''(t)| < 4 \quad \forall\ t \in [0,1]$$

Pelo TVM,

$$f'(x) - f'(y) = f''(z)(x-y)$$

$$\Rightarrow |f'(x) - f'(y)| < 4\,|x-y|$$

Como o movimento começa e termina em repouso, segue que em $t=0$ e $t=1$ a velocidade é nula, ou seja,

$$f'(0) = f'(1) = 0$$

Assim,

$$t \in [0, 1/2] \Rightarrow |f'(t)| = |f'(t) - f'(0)| < 4t$$

$$t \in [1/2, 1] \Rightarrow |f'(t)| = |f'(1) - f'(t)| < 4(1-t)$$

Logo, a distância percorrida, $f(1) - f(0)$, satisfaz

$$f(1) - f(0) = \int_0^1 f'(t)\,dt \leq \int_0^1 |f'(t)|\,dt$$

$$< \int_0^{1/2} 4t\,dt + \int_{1/2}^1 4(1-t)\,dt$$

$$= \left.\frac{4t^2}{2}\right|_0^{1/2} - \left.\frac{4(1-t)^2}{2}\right|_{1/2}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

Ou seja a distância percorrida teria sido inferior a 1, absurdo. Logo, é necessário que em algum momento a aceleração tenha sido de 4 ou mais.

Exercício 17. Suponha que f tem primeira e segunda derivadas em toda a reta real. Prove que se f for sempre positiva e côncava em todo ponto, então f é constante.

Se f não fosse constante, haveria $c \in \mathbb{R}$ tal que $f'(c) \neq 0$. Assim, para qualquer $x \in \mathbb{R}$, o fato de que $f''(x) \leq 0$ implica

$$f'(x) - f'(c) = \int_c^x f''(t)\,dt \leq \int_c^x 0\,dt = 0$$

$$\Rightarrow f'(x) \leq f'(c) \quad \forall x \in \mathbb{R}$$

Assim,

$$f(x) - f(0) = \int_0^x f'(t)\,dt \leq \int_0^x f'(c)\,dt = x f'(c)$$

Logo,

$$f(x) \leq f'(c) \cdot x + f(0)$$

Como $f'(c) \neq 0$, o lado direito tende a $-\infty$ quando $x \to -\infty$. Isso implica que $f < 0$ em algum ponto, uma contradição com o enunciado.

Logo, é preciso que $f'(x) = 0 \ \forall x \in \mathbb{R}$, e daí f é constante.

Exercício 1. Calcule a integral fazendo a substituição sugerida:

(a) $\int \cos 2x\, dx,\ u = 2x$

(b) $\int xe^{-x^2}\, dx,\ u = -x^2$

(c) $\int x^2\sqrt{x^3+1}\, dx,\ u = x^3+1$

(d) $\int \operatorname{sen}^2\theta \cos\theta\, d\theta,\ u = \operatorname{sen}\theta$

(e) $\int \frac{x^3}{x^4-5}\, dx,\ u = x^4-5$

(f) $\int \sqrt{2t+1}\, dt,\ u = 2t+1$

(a) $u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2}du$

$$\int \cos 2x\, dx = \int \frac{1}{2}\cos u\, du = \frac{\operatorname{sen} u}{2} = \frac{\operatorname{sen} 2x}{2}$$

(b) $u = -x^2 \Rightarrow du = -2x\, dx \Rightarrow x\,dx = -\frac{1}{2}du$

$$\int xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$$

(c) $u = x^3+1 \Rightarrow du = 3x^2dx \Rightarrow x^2dx = \frac{1}{3}du$

$$\int x^2\sqrt{x^3+1}\,dx = \int \frac{1}{3}u^{1/2}du = \frac{2}{9}u^{3/2}$$

$$= \frac{2}{9}(x^3+1)^{3/2}$$

(d) $u = \operatorname{sen}\theta \Rightarrow du = \cos\theta\, d\theta$

$$\int \operatorname{sen}^2\theta \cos\theta\, d\theta = \int u^2 du = \frac{1}{3}u^3$$

(e) $u = x^4-5 \Rightarrow du = 4x^3dx \Rightarrow x^3dx = \frac{1}{4}du$

$$\int \frac{x^3}{x^4-5}dx = \int \frac{1}{4}\frac{du}{u} = \frac{1}{4}\log u = \frac{1}{4}\log(x^4-5)$$

(f) $u = 2t+1 \Rightarrow du = 2dt \Rightarrow dt = \frac{1}{2}du$

$$\int \sqrt{2t+1}\,dt = \int \frac{1}{2}u^{1/2}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{(2t+1)^{3/2}}{3}$$

Exercício 2. Calcule a integral indefinida:

(a) $\int x\sqrt{1-x^2}\,dx$ (b) $\int (3x-2)^{20}\,dx$ (c) $\int \cos(\pi t/2)\,dt$

(d) $\int \operatorname{sen}\pi t\,dt$ (e) $\int \cos^3\theta \operatorname{sen}\theta\,d\theta$ (f) $\int \frac{e^u}{(1-e^u)^2}\,du$

(g) $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$ (h) $\int \frac{(\log x)^2}{x}\,dx$ (i) $\int \cos^4\theta \operatorname{sen}\theta\,d\theta$

(j) $\int x^2 e^{x^3}\,dx$ (k) $\int \operatorname{sen} t\sqrt{1+\cos t}\,dt$ (l) $\int \sec^2 2\theta\,d\theta$

(m) $\int y^2(4-y^3)^{2/3}\,dy$ (n) $\int e^{-5r}\,dr$ (o) $\int \frac{z^2}{z^3+1}\,dz$

(p) $\int \operatorname{sen} x \operatorname{sen}(\cos x)\,dx$ (q) $\int x\sqrt{x+2}\,dx$ (r) $\int \frac{dx}{ax+b}$

(s) $\int (x^2+1)(x^3+3x)^4\,dx$ (t) $\int 5^t \operatorname{sen} 5^t\,dt$ (u) $\int \frac{\sec^2 x}{\tan^2 x}\,dx$

(v) $\int e^{\cos t}\operatorname{sen} t\,dt$ (w) $\int \frac{x}{x^2+4}\,dx$ (x) $\int \frac{\cos(\pi/x)}{x^2}\,dx$

(a) $u = 1-x^2 \Rightarrow du = -2x\,dx \Rightarrow x\,dx = -\frac{1}{2}du$

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}$$

(b) $u = 3x-2 \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}du$

$$\int (3x-2)^{20}\,dx = \int \frac{1}{3}u^{20}\,du = \frac{1}{3}\cdot\frac{1}{21}\cdot u^{21}$$

$$= \frac{1}{63}(3x-2)^{21}$$

(c) $u = \frac{\pi t}{2} \Rightarrow t = \frac{2u}{\pi} \Rightarrow dt = \frac{2}{\pi}du$

$$\int \cos\frac{\pi t}{2}\,dt = \int \frac{2}{\pi}\cos u\,du = \frac{2}{\pi}\operatorname{sen} u$$

$$= \frac{2}{\pi}\operatorname{sen}\frac{\pi t}{2}$$

(d) $u = \pi t \Rightarrow du = \pi dt \Rightarrow dt = \frac{1}{\pi} du$

$$\int \operatorname{sen} \pi t \, dt = \int \frac{1}{\pi} \operatorname{sen} u \, du = -\frac{1}{\pi} \cos u = -\frac{1}{\pi} \cos \pi t$$

(e) $u = \cos\theta \Rightarrow du = -\operatorname{sen}\theta \, d\theta$

$$\int \cos^3\theta \operatorname{sen}\theta \, d\theta = -\int u^3 \, du = -\frac{1}{4} u^4 = -\frac{1}{4} \cos^4\theta$$

(f) $x = 1 - e^u \Rightarrow dx = -e^u du$

$$\int \frac{e^u}{(1-e^u)^2} du = -\int \frac{dx}{x^2} = x^{-1} = \frac{1}{1-e^u}$$

(g) $u = 3ax + bx^3 \Rightarrow du = 3(a + bx^2) dx$

$$\int \frac{a + bx^2}{(3ax + bx^3)^{1/2}} dx = \frac{1}{3} \int \frac{du}{u^{1/2}} = \frac{2}{3} u^{1/2}$$

$$= \frac{2}{3} \sqrt{3ax + bx^3}$$

(h) $u = \log x \Rightarrow du = \frac{dx}{x}$

$$\int \frac{(\log x)^2}{x} dx = \int u^2 du = \frac{1}{3} u^3 = \frac{1}{3} (\log x)^3$$

(i) $u = \cos\theta \Rightarrow -du = \operatorname{sen}\theta\, d\theta$

$$\int \cos^4\theta \operatorname{sen}\theta\, d\theta = -\int u^4\, du = -\frac{1}{5}u^5$$

$$= -\frac{1}{5}\cos^5\theta$$

(j) $u = e^{x^3} \Rightarrow du = 3x^2 e^{x^3} dx$

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int du = \frac{1}{3}u = \frac{1}{3}e^{x^3}$$

(k) $u = 1 + \cos t \Rightarrow du = -\operatorname{sen} t\, dt$

$$\int \operatorname{sen} t \sqrt{1+\cos t}\, dt = -\int u^{1/2}\, du = -\frac{2}{3}u^{3/2}$$

$$= -\frac{2}{3}(1+\cos t)^{3/2}$$

(ℓ) $u = 2\theta \Rightarrow du = 2\, d\theta$

$$\int \sec^2 2\theta\, d\theta = \frac{1}{2}\int \sec^2 u\, du = \frac{1}{2}\tan u$$

$$= \frac{1}{2}\tan 2\theta$$

(m) $u = 4 - y^3 \Rightarrow du = -3y^2\, dy$

$$\int y^2(4-y^3)^{2/3}\,dy = -\frac{1}{3}\int u^{2/3}\,du = -\frac{1}{3}\cdot\frac{3}{5}u^{5/3}$$
$$= -\frac{1}{5}(4-y^3)^{5/3}$$

(n) $u = -5r \Rightarrow du = -5dr$

$$\int e^{-5r}dr = -\frac{1}{5}\int e^u du = -\frac{1}{5}e^u = -\frac{1}{5}e^{-5r}$$

(o) $u = z^3+1 \Rightarrow du = 3z^2dz$

$$\int \frac{z^2}{z^3+1}dz = \frac{1}{3}\int\frac{du}{u} = \frac{1}{3}\log u = \frac{1}{3}\log(z^3+1)$$

(p) $u = \cos x \Rightarrow du = -\operatorname{sen} x\,dx$

$$\int \operatorname{sen} x\,\operatorname{sen}(\cos x)\,dx = -\int \operatorname{sen} u\,du = \cos u$$
$$= \cos(\cos x)$$

(q) $u = x+2 \Rightarrow x = u-2 \Rightarrow dx = du$

$$\int x\sqrt{x+2}\,dx = \int (u-2)u^{1/2}du = \int u^{3/2} - 2u^{1/2}\,du$$
$$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2}$$

(r) $u = ax+b \Rightarrow du = a\,dx$

$$\int\frac{dx}{ax+b} = \int\frac{1}{a}\frac{du}{u} = \frac{1}{a}\log u = \frac{1}{a}\log(ax+b)$$

(s) $u = x^3 + 3x \Rightarrow du = 3(x^2+1)\,dx$

$$\int (x^2+1)(x^3+3x)^4\,dx = \int \frac{1}{3} u^4\,du = \frac{1}{15}u^5$$

$$= \frac{1}{15}(x^3+3x)^5$$

(t) $u = 5^t = e^{t\log 5} \Rightarrow du = \log 5 \cdot 5^t$

$$\int 5^t \operatorname{sen} 5^t\,dt = \frac{1}{\log 5}\int \operatorname{sen} u\,du = \frac{-1}{\log 5}\cos u$$

$$= -\frac{1}{\log 5}\cos 5^t$$

(u) $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \frac{du}{u^2} = -\frac{1}{u} = -\cot x$$

(v) $u = \cos t \Rightarrow du = -\operatorname{sen} t\,dt$

$$\int e^{\cos t}\operatorname{sen} t\,dt = -\int e^u\,du = -e^u = -e^{\cos t}$$

(w) $u = x^2 + 4 \Rightarrow du = 2x\,dx$

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log u = \frac{1}{2}\log(x^2+4)$$

(x) $u = \pi/x \Rightarrow du = (-\pi/x^2)\,dx$

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\operatorname{sen} u$$

$$= -\frac{1}{\pi}\operatorname{sen}\left(\frac{\pi}{x}\right)$$

Exercício 3. Calcule:

(a) $\displaystyle\int \frac{2^t}{2^t+3}\,dt$ (b) $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$ (c) $\displaystyle\int \operatorname{senh}^2 x\cosh x\,dx$

(d) $\displaystyle\int \frac{\cos(\log t)}{t}\,dt$ (e) $\displaystyle\int \frac{x}{1+x^4}\,dx$ (f) $\displaystyle\int \frac{\operatorname{sen} 2x}{1+\cos^2 x}\,dx$

(g) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\operatorname{arcsen} x}$ (h) $\displaystyle\int \frac{1+x}{1+x^2}\,dx$ (i) $\displaystyle\int x(2x+5)^8\,dx$

(j) $\displaystyle\int x^2\sqrt{2+x}\,dx$ (k) $\displaystyle\int x^3\sqrt{x^2+1}\,dx$ (l) $\displaystyle\int \tan^2\theta\sec^2\theta\,d\theta$

(a) $u = 2^t+3 \Rightarrow du = \log 2\cdot 2^t\,dt$

$$\int \frac{2^t}{2^t+3}dt = \frac{1}{\log 2}\int \frac{du}{u} = \frac{1}{\log 2}\log u = \frac{1}{\log 2}\log(2^t+3)$$

(b) $u = 1+\tan t \Rightarrow du = \sec^2 t\,dt = \dfrac{dt}{\cos^2 t}$

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} = \int \frac{du}{u^{1/2}} = 2u^{1/2}$$

$$= 2\sqrt{1+\tan t}$$

(c) $u = \operatorname{senh} x \Rightarrow du = \cosh x\,dx$

$$\int \operatorname{senh}^2 x\cosh x\,dx = \int u^2\,du = \frac{1}{3}u^3$$

$$= \frac{1}{3}\operatorname{senh}^3 x$$

(d) $u = \log t \Rightarrow du = \frac{1}{t}dt$

$$\int \frac{\cos(\log t)}{t}dt = \int \cos u\,du = \operatorname{sen} u$$

$$= \operatorname{sen}(\log t)$$

(e) $u = x^2 \Rightarrow du = 2x\,dx$

$$\int \frac{x}{1+x^4}dx = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan u$$
$$= \frac{1}{2}\arctan x^2$$

(f) $\operatorname{sen} 2x = 2\operatorname{sen} x \cdot \cos x$

$u = 1+\cos^2 x \Rightarrow du = -2\cos x \operatorname{sen} x\,dx$
$= -\operatorname{sen} 2x\,dx$

$$\int \frac{\operatorname{sen} 2x}{1+\cos^2 x}dx = -\int \frac{du}{u} = -\log u$$
$$= -\log(1+\cos^2 x)$$

(g) $u = \operatorname{arcsen} x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$

$$\int \frac{dx}{\sqrt{1-x^2}\operatorname{arcsen} x} = \int \frac{du}{u} = \log u = \log(\operatorname{arcsen} x)$$

(h) $u = 1+x^2 \Rightarrow du = 2x\,dx$

$$\int \frac{1+x}{1+x^2}dx = \int \frac{dx}{1+x^2} + \int \frac{x\,dx}{1+x^2}$$
$$= \arctan x + \frac{1}{2}\int \frac{du}{u} = \arctan x + \frac{1}{2}\log u$$
$$= \arctan x + \frac{1}{2}\log(1+x^2)$$

(i) $u = 2x+5 \Rightarrow x = \frac{1}{2}(u-5) \Rightarrow dx = \frac{1}{2}du$

$$\int x(2x+5)^8\,dx = \int \frac{1}{2}(u-5)\,u^8\,du$$

$$= \frac{1}{2}\int u^9 - 5u^8\,du = \frac{1}{2}\left(\frac{u^{10}}{10} - \frac{5}{9}u^9\right)$$

$$= \frac{1}{2}(2x+5)^9\cdot\left(\frac{2x+5}{10} - \frac{5}{9}\right)$$

$$= \frac{1}{2}(2x+5)^9\cdot\frac{(18x-5)}{90} = \frac{1}{180}(2x+5)^9\cdot(18x-5)$$

(j) $u = 2+x \Rightarrow x = u-2 \Rightarrow dx = du$

$$\int x^2\sqrt{2+x}\,dx = \int (u-2)^2 u\,du = \int u^3 - 4u^2 + 4u\,du$$

$$= \frac{u^4}{4} - \frac{4}{3}u^3 + 2u^2 = (2+x)^2\left[\frac{(2+x)^2}{4} - \frac{4}{3}(2+x) + 2\right]$$

(k) $u = x^2+1 \Rightarrow du = 2x\,dx$

$$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{2}\int (u-1)\,u^{1/2}\,du$$

$$= \frac{1}{2}\int u^{3/2} - u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)$$

$$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2}$$

(ℓ) $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$

$$\int \tan^2\theta\,\sec^2\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\tan^3\theta$$

Exercício 4. Calcule as seguintes integrais:

(a) $\displaystyle\int_0^1 \cos(\pi t/2)\,dt$ (b) $\displaystyle\int_0^1 \sqrt[3]{1+7x}\,dx$ (c) $\displaystyle\int_0^{\pi/6} \frac{\operatorname{sen} t}{\cos^2 t}\,dt$

(d) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$ (e) $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 + x^4 \tan x\,dx$ (f) $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$

(g) $\displaystyle\int_0^a x\sqrt{x^2+a^2}\,dx$ (h) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\log x}}$ (i) $\displaystyle\int_0^{1/2} \frac{\operatorname{arcsen} x}{\sqrt{1-x^2}}\,dx$

(j) $\displaystyle\int_{\pi/3}^{2\pi/3} \operatorname{cossec}^2(t/2)\,dt$ (k) $\displaystyle\int_0^{\pi/2} \cos x \operatorname{sen}(\operatorname{sen} x)\,dx$ (l) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

(a) $u = \frac{\pi t}{2} \Rightarrow dt = \frac{2}{\pi}du$

$$\int_0^1 \cos\frac{\pi t}{2}\,dt = \int_0^{\pi/2} \frac{2}{\pi}\cos u\,du = \frac{2}{\pi}\operatorname{sen} u\Big|_0^{\pi/2} = \frac{2}{\pi}$$

(b) $u = 1+7x \Rightarrow du = 7dx$

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 \frac{1}{7}u^{1/3}du = \frac{1}{7}\cdot\frac{3}{4}u^{4/3}\Big|_1^8$$

$$= \frac{3}{7\cdot 4}\cdot(16-1) = \frac{45}{28}$$

(c) $u = \cos t \Rightarrow du = -\operatorname{sen} t\,dt$

$$\int_0^{\pi/6} \frac{\operatorname{sen} t}{\cos^2 t}\,dt = -\int_1^{\sqrt{3}/2} \frac{du}{u^2} = \frac{1}{u}\Big|_1^{\sqrt{3}/2} = \frac{2}{\sqrt{3}} - 1$$

(d) $u = 1/x \Rightarrow du = -\frac{1}{x^2}dx$

$$\int_1^2 \frac{e^{1/x}}{x^2}dx = -\int_1^{1/2} e^u du = e^u\Big|_{1/2}^1 = e - \sqrt{e}$$

(e) $f(x) = x^3 + x^4\tan x$ é Ímpar. $(f(-x) = -f(x))$.

Logo,

$$\int_{-\pi/4}^{\pi/4} f(x)\,dx = \int_{-\pi/4}^{0} f(x)\,dx + \int_{0}^{\pi/4} f(x)\,dx$$

$u = -x \Rightarrow du = -dx$

$$= -\int_{\pi/4}^{0} f(-u)\,du + \int_{0}^{\pi/4} f(x)\,dx$$

$$= \int_{\pi/4}^{0} f(u)\,du + \int_{0}^{\pi/4} f(x)\,dx$$

$$= -\int_{0}^{\pi/4} f(u)\,du - \int_{0}^{\pi/4} f(x)\,dx = 0$$

(Lembre que a letra não importa!)

(f) $u = x - 1 \Rightarrow du = dx$

$$\int_{1}^{2} x\sqrt{x-1}\,dx = \int_{0}^{1} (u+1)\,u^{1/2}\,du = \int_{0}^{1} u^{3/2} + u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Big|_{0}^{1} = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

(g) $u = x^2 + a^2 \Rightarrow du = 2x\,dx$

$$\int_{0}^{a} x\sqrt{x^2+a^2}\,dx = \int_{a^2}^{2a^2} \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{a^2}^{2a^2}$$

$$= \frac{1}{3}\left(8a^3 - a^3\right) = \frac{7a^3}{3}$$

(h) $u = \log x \Rightarrow du = \frac{dx}{x}$

$$\int_e^{e^4} \frac{dx}{x\sqrt{\log x}} = \int_1^4 \frac{du}{u^{1/2}} = 2u^{1/2}\Big|_1^4 = 2$$

(i) $u = \arcsen x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$

$$\int_0^{1/2} \frac{\arcsen x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \frac{1}{2}u^2\Big|_0^{\pi/6} = \frac{\pi^2}{72}$$

(j) $u = t/2 \Rightarrow du = \frac{1}{2}dt$

$$\int_{\pi/3}^{2\pi/3} \operatorname{cossec}^2 \frac{t}{2}\,dt = 2\int_{\pi/6}^{\pi/3} \operatorname{cossec}^2 u\,du =$$

$$= -2\cot u\Big|_{\pi/6}^{\pi/3} = -2\cdot\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = \frac{4}{\sqrt{3}}$$

(k) $u = \operatorname{sen} x \Rightarrow du = \cos x\,dx$

$$\int_0^{\pi/2} \cos x\,\operatorname{sen}(\operatorname{sen} x)\,dx = \int_0^1 \operatorname{sen} u\,du = -\cos u\Big|_0^1$$

$$= 1 - \cos(1)$$

(l) $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx$

$\Rightarrow dx = 2(u-1)\,du$

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{2(u-1)}{u^4}\,du = 2\int_1^2 u^{-3} - u^{-4}\,du$$

$$= 2\cdot\left(-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right)\Big|_1^2 = 2\cdot\left(-\frac{1}{8} + \frac{1}{24} + \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{2}{24}\left(-3+1+12-8\right) = \frac{4}{24} = \frac{1}{6}$$

Exercício 5. Mostre que

$$0 \leqslant \int_{-2}^{3} \operatorname{sen} \sqrt[3]{x}\, dx \leqslant 1$$

Note que $\operatorname{sen} \sqrt[3]{x}$ é uma função ímpar:

$$\operatorname{sen} \sqrt[3]{(-x)} = \operatorname{sen}(-\sqrt[3]{x}) = -\operatorname{sen} \sqrt[3]{x}$$

Logo,

$$\int_{-2}^{2} \operatorname{sen} \sqrt[3]{x}\, dx = 0$$

Daí,

$$\int_{-2}^{3} \operatorname{sen} \sqrt[3]{x}\, dx = \int_{2}^{3} \operatorname{sen} \sqrt[3]{x}\, dx$$

Em $2 \leq x \leq 3$, temos

$$0 \leq \operatorname{sen} \sqrt[3]{x} \leq 1$$

pois $0 < \sqrt[3]{2} < \sqrt[3]{3} < 3 < \pi$

Logo,

$$0 = \int_{2}^{3} 0\, dx \leq \int_{2}^{3} \operatorname{sen} \sqrt[3]{x}\, dx \leq \int_{2}^{3} 1\, dx = 1$$

e então

$$0 \leq \int_{-2}^{3} \operatorname{sen} \sqrt[3]{x}\, dx \leq 1.$$

Exercício 6. Se f é contínua em $[0,\pi]$, use a substituição $u = \pi - x$ para demonstrar que

$$\int_0^{\pi} x f(\operatorname{sen} x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\operatorname{sen} x)\,dx$$

$$u = \pi - x \Rightarrow x = \pi - u \Rightarrow dx = -du$$

$$\int_0^{\pi} x f(\operatorname{sen} x)\,dx = -\int_{\pi}^{0} (\pi - u) f[\operatorname{sen}(\pi - u)]\,du$$

$$\operatorname{sen}(\pi - u) = \operatorname{sen} u$$

$$= \int_0^{\pi} (\pi - u) f(\operatorname{sen} u)\,du$$

$$= \pi \int_0^{\pi} f(\operatorname{sen} u)\,du - \int_0^{\pi} u f(\operatorname{sen} u)\,du$$

Lembrando que a letra não importa, segue que

$$2\int_0^{\pi} x f(\operatorname{sen} x)\,dx = \pi \int_0^{\pi} f(\operatorname{sen} x)\,dx$$

$$\Rightarrow \int_0^{\pi} x f(\operatorname{sen} x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\operatorname{sen} x)\,dx$$

Exercício 7. Use o exercício anterior para calcular

$$\int_0^\pi \frac{x \operatorname{sen} x}{1+\cos^2 x}\,dx$$

$$\frac{\operatorname{sen} x}{1+\cos^2 x} = \frac{\operatorname{sen} x}{2-\operatorname{sen}^2 x} = f(\operatorname{sen} x),$$

onde

$$f(u) = \frac{u}{2-u^2}$$

Pelo exercício anterior,

$$\int_0^\pi x f(\operatorname{sen} x)\,dx = \frac{\pi}{2}\int_0^\pi f(\operatorname{sen} x)\,dx$$

$$= \frac{\pi}{2}\int_0^\pi \frac{\operatorname{sen} x}{1+\cos^2 x}\,dx$$

$$u = \cos x \Rightarrow du = -\operatorname{sen} x\,dx$$

$$= -\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^{1}\frac{du}{1+u^2} = \frac{\pi}{2}\arctan u\Big|_{-1}^{1}$$

$$= \frac{\pi}{2}\cdot\frac{2\pi}{4} = \frac{\pi^2}{4}$$

Exercício 8. Calcule $\int_0^1 \sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}\,dx$

Vamos avaliar $\int_0^1 \sqrt[3]{1-x^7}\,dx$:

$$u = (1-x^7)^{1/3} \Rightarrow du = \frac{1}{3}(1-x^7)^{-2/3} \cdot -7x^6\,dx$$
$$= \frac{-7}{3u^2}x^6\,dx$$

$$\Rightarrow 1-x^7 = u^3$$
$$\Rightarrow x^7 = 1-u^3 \Rightarrow x = \sqrt[7]{1-u^3}$$
$$\Rightarrow x^6 = (1-u^3)^{6/7}$$
$$\Rightarrow du = -\frac{7}{3u^2}\cdot(1-u^3)^{6/7}\,dx$$
$$\Rightarrow dx = -\frac{3}{7}u^2(1-u^3)^{-6/7}\,du$$

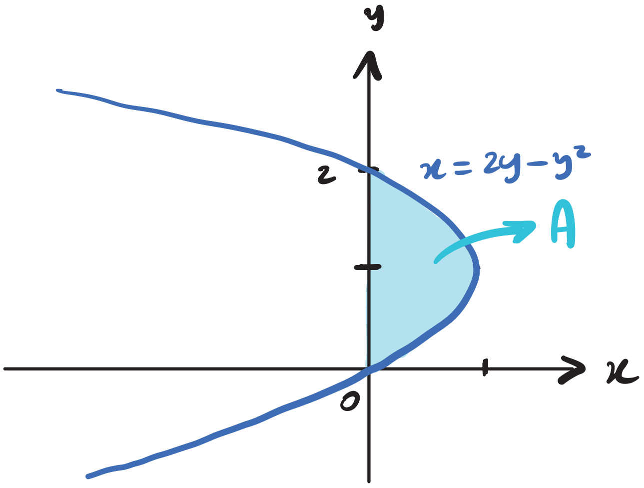
Assim,

$$\int_0^1 \sqrt[3]{1-x^7}\,dx = \int_0^1 \frac{3}{7}u \cdot u^2(1-u^3)^{-6/7}\,du$$
$$= -\int_0^1 u \cdot -\frac{3}{7}u^2(1-u^3)^{-6/7}\,du$$
$$= -\int_0^1 u \cdot \frac{d}{du}(1-u^3)^{1/7}\,du$$
$$= \int_0^1 \sqrt[7]{1-u^3}\,du$$

Portanto,

$$\int_0^1 \sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}\,dx = 0$$

Exercício 1. Esboce a região entre a curva $x = 2y - y^2$ e o eixo y e calcule a sua área.



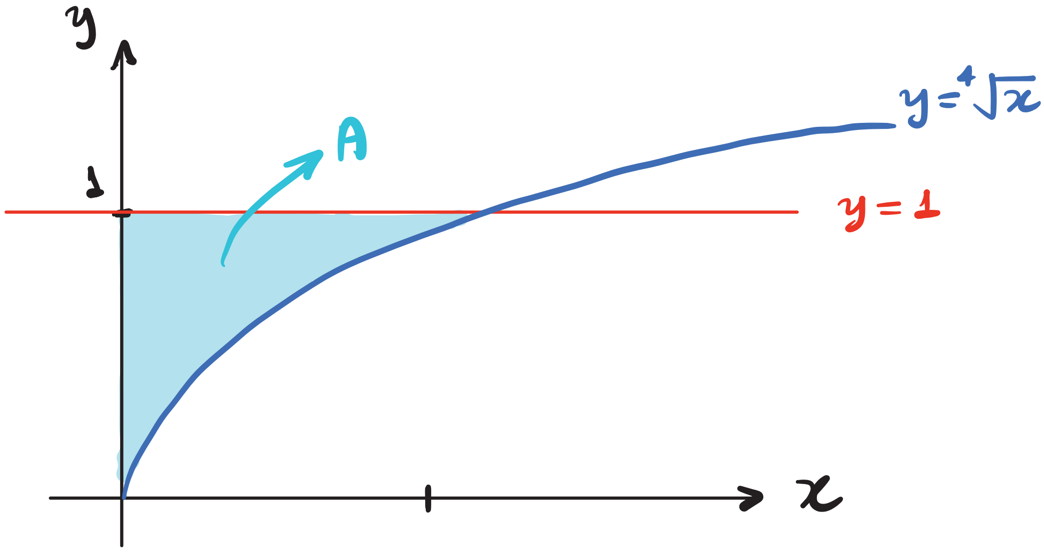
$$x = -y(y-2)$$

$$A = \int_0^2 2y - y^2 \, dy$$

$$= y^2 - \frac{y^3}{3} \Big|_0^2$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

Exercício 2. Esboce a região entre as curvas $y = 1$, $y = \sqrt[4]{x}$ e o eixo y e calcule a sua área.



$$y = \sqrt[4]{x} \iff x = y^4$$

$$A = \int_0^1 y^4 \, dy = \left. \frac{y^5}{5} \right|_0^1 = \frac{1}{5}$$

Exercício 3. Esboce a região entre as curvas e calcule sua área:

(a) $y = 1/x, y = \sqrt[3]{x}$ e $x = 8$

(b) $y = e^x, y = xe^{x^2}$ e $x = 0$

(c) $x = y^2 - 2, x = e^y, y = 1$ e $y = -1$

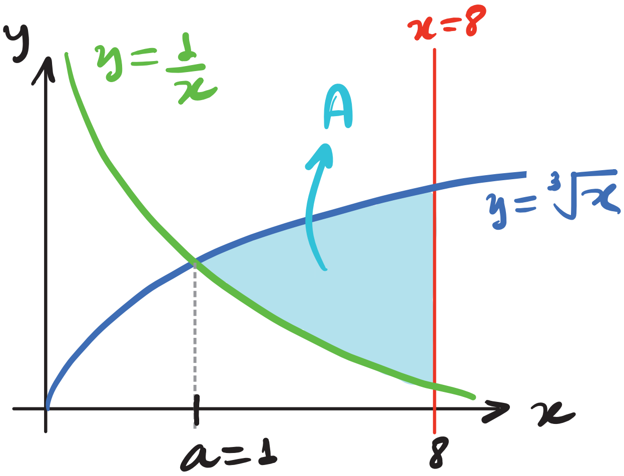
(d) $y = \sec^2 x, y = 8\cos x, \ -\pi/3 \leqslant x \leqslant \pi/3$

(e) $y = \sqrt{x-1}$ e $x - y = 1$

(f) $y = \cos \pi x$ e $y = 4x^2 - 1$

(g) $y = \operatorname{senh} x, y = e^x, x = 0$ e $x = 2$

(a)



Interseção em a:

$$\frac{1}{a} = \sqrt[3]{a}$$

$$\Rightarrow 1 = a^3 \cdot a$$

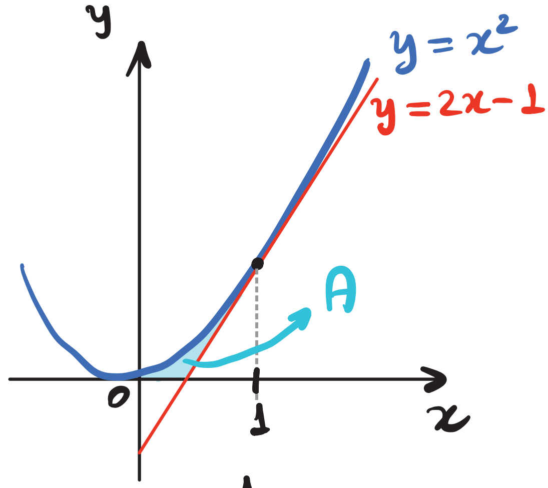
$$\Rightarrow a^4 = 1$$

$$\Rightarrow a = 1$$

$$A = \int_1^8 x^{1/3} - \frac{1}{x}\, dx = \frac{3}{4}x^{4/3} - \log x \Big|_1^8$$

$$= \frac{3}{4} \cdot 16 - \frac{3}{4} - 3\log 2 = \frac{45}{4} - 3\log 2$$

Exercício 4. Encontre a área da região delimitada pela parábola $y = x^2$, pela reta tangente à parábola no ponto $(1, 1)$ e pelo eixo x.



$y'(x) = 2x \Rightarrow y'(1) = 2$

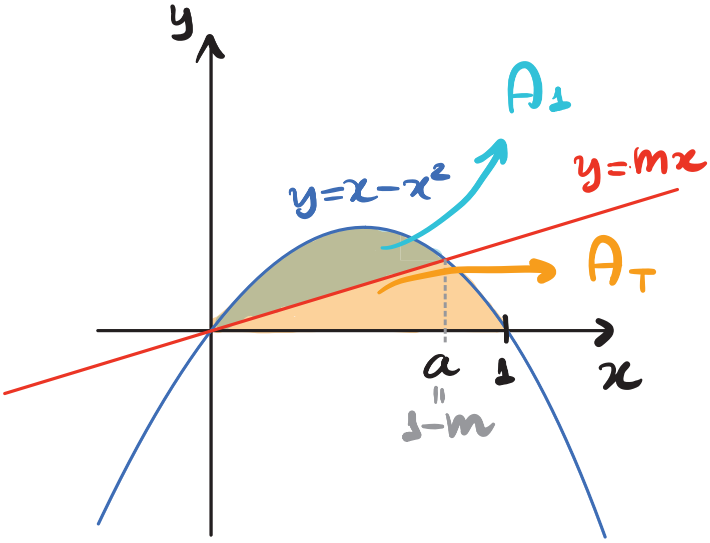
Tangente:

$$y = 1 + y'(1)(x-1)$$
$$= 1 + 2x - 2$$
$$= 2x - 1$$

$$A = \int_0^1 x^2 - (2x-1)\,dx = \left.\frac{x^3}{3} - x^2 + x\right|_0^1$$

$$= \frac{1}{3} - 1 + 1 = \frac{1}{3}$$

Exercício 5. Existe uma reta que passa pela origem e que divide a região delimitada pela parábola $y = x - x^2$ e o eixo x em duas regiões de áreas iguais. Qual é a inclinação dessa reta?



Reta procurada:

$y = mx$ ↳ inclinação

Interseção $a>0$:

$ma = a - a^2$

$\Rightarrow a = 1 - m$

Área total da parábola:

$$A_T = \int_0^1 x - x^2\, dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}$$

Área topo procurada:

$$A_1 = \int_0^a x - x^2 - mx\, dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} - \frac{mx^2}{2}\Big|_0^{1-m}$$

$$= \frac{(1-m)}{2}(1-m)^2 - \frac{(1-m)^3}{3} = \frac{(1-m)^3}{6}$$

A condição é ter $A_1 = \frac{1}{2} A_T$

$$\Rightarrow \frac{(1-m)^3}{6} = \frac{1}{2}\cdot\frac{1}{6} \Rightarrow (1-m) = \frac{1}{\sqrt[3]{2}}$$

$$\Rightarrow m = \frac{\sqrt[3]{2} - 1}{\sqrt[3]{2}}$$

Exercício 6. Encontre o volume do sólido obtido pela rotação da região delimitada pelas curvas dadas em torno das retas especificadas.

(a) $y = x + 1, \quad y = 0, \quad x = 0, \quad x = 2$ em torno do eixo x

(b) $y = \sqrt{x-1}, \quad y = 0, \quad x = 5$ em torno do eixo x

(c) $y = e^x, \quad y = 0, \quad x = -1, \quad x = 1$ em torno do eixo x

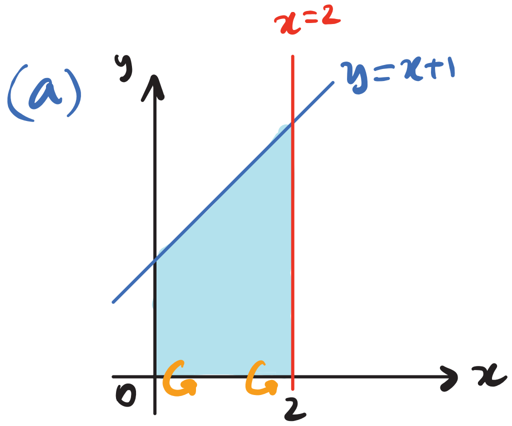
(d) $y = \log x, \quad y = 1, \quad y = 2, \quad x = 0$ em torno do eixo y

(e) $x = 2 - y^2, \quad x = y^4$ em torno do eixo y

(f) $y = x^3, \quad y = 1, \quad x = 2$ em torno de $y = -3$

(g) $y = \operatorname{sen} x, \quad y = \cos x, \quad 0 \leqslant x \leqslant \pi/4$ em torno de $y = -1$

(h) $y = x, \quad y = \sqrt{x}$ em torno de $x = 2$

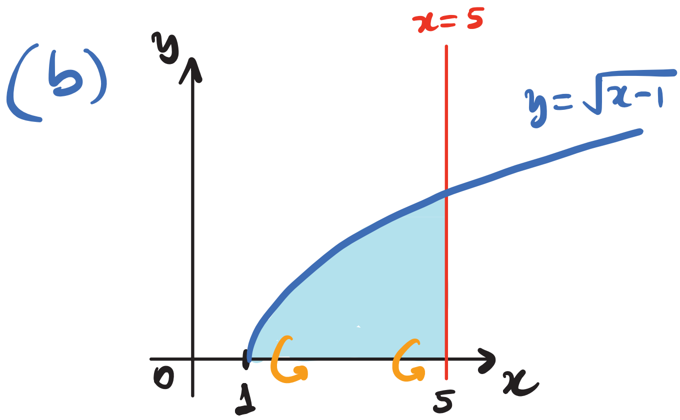


$$V = \int_0^2 \pi (x+1)^2 \, dx$$

$$= \int_0^2 \pi (x^2 + 2x + 1) \, dx$$

$$= \pi \left(\frac{x^3}{3} + x^2 + x \right) \Big|_0^2$$

$$= \pi \left(\frac{8}{3} + 4 + 2 \right) = \frac{26\pi}{3}$$



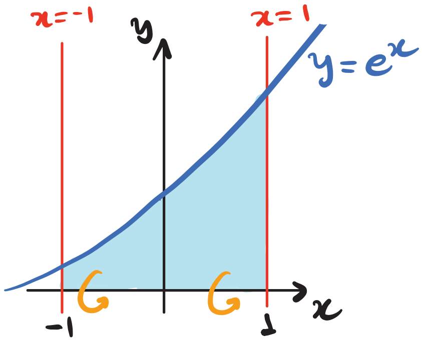
$$V = \int_1^5 \pi (x-1) \, dx$$

$$= \pi \left(\frac{x^2}{2} - x \right) \Big|_1^5$$

$$= \pi \left(\frac{25}{2} - \frac{10}{2} - \frac{1}{2} + \frac{2}{2} \right)$$

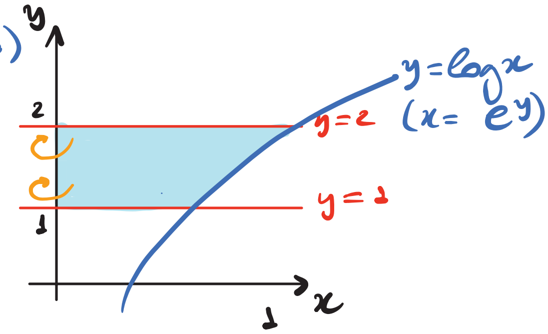
$$= 8\pi$$

(c)



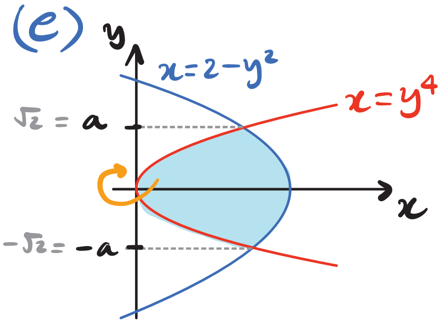
$$V = \int_{-1}^{1} \pi e^{2x}\, dx = \pi \frac{e^{2x}}{2}\Big|_{-1}^{1} = \pi\left(\frac{e^2 - e^{-2}}{2}\right)$$

(d)



$$V = \int_1^2 \pi e^{2y}\, dy = \pi \frac{e^{2y}}{2}\Big|_1^2 = \pi\left(\frac{e^4 - e^2}{2}\right)$$

(e)



Interseções $a, -a$:

$a^4 = 2 - a^2$

$\Rightarrow a^4 - a^2 - 2 = 0$

$a^2 = \frac{1 \pm 3}{2} = 2 \quad (a^2 \geq 0)$

$\Rightarrow a = \pm\sqrt{2}$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left[(2 - y^2)^2 - (y^4)^2\right] dy$$

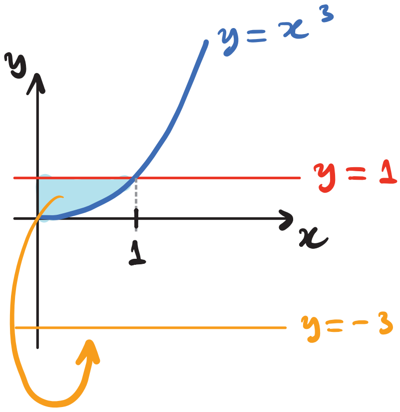
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left(4 - 4y^2 + y^4 - y^8\right) dy$$

$$= \pi\left(4y - \frac{4}{3}y^3 + \frac{y^5}{5} - \frac{y^9}{9}\right)\Bigg|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= 2\sqrt{2}\,\pi\left(4 - \frac{4}{3}\cdot 2 + \frac{4}{5} - \frac{16}{9}\right)$$

$$= \frac{2\sqrt{2}\,\pi}{45}\left(180 - 120 + 36 - 80\right) = \frac{32\sqrt{2}\,\pi}{45}$$

(f)



$$V = \int_0^1 \pi\left[4^2 - (3 + x^3)^2\right] dx$$

$$= 16\pi - \pi\int_0^1 9 + 6x^3 + x^6\, dx$$

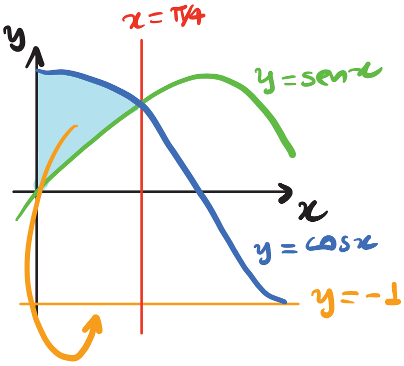
$$= 16\pi - \pi\left(9x + \frac{6x^4}{4} + \frac{x^7}{7}\right)\Big|_0^1$$

$$= 16\pi - \pi\left(9 + \frac{3}{2} + \frac{1}{7}\right)$$

$$\Rightarrow V = 16\pi - \frac{(126 + 21 + 2)}{14}\pi$$

$$= \frac{75\pi}{14}$$

(g)



$$V = \int_0^{\pi/4} \pi\left[(1 + \cos x)^2 - (1 + \text{sen}\, x)^2\right] dx$$

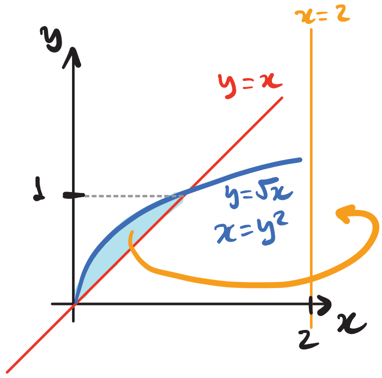
$$= \pi\int_0^{\pi/4} 2(\cos x - \text{sen}\, x) + \cos^2 x - \text{sen}^2 x\, dx$$

$$= \pi\int_0^{\pi/4} 2(\cos x - \text{sen}\, x) + \cos 2x\, dx$$

$$= \pi\left(2\,\text{sen}\,x + 2\cos x + \frac{\text{sen}\,2x}{2}\right)\Bigg|_0^{\pi/4}$$

$$= \pi\left(\sqrt{2} + \sqrt{2} - 2 + \frac{1}{2}\right) = \left(2\sqrt{2} - \frac{3}{2}\right)\pi$$

(h)



$$V = \int_0^1 \pi\left[(2-y^2)^2 - (2-y)^2\right] dy$$

$$= \pi\int_0^1 \left[\cancel{4} - 4y^2 + y^4 - \cancel{4} + 4y - y^2\right] dy$$

$$= \pi\int_0^1 y^4 - 5y^2 + 4y\; dy$$

$$\Rightarrow V = \pi\left(\frac{y^5}{5} - \frac{5}{3}y^3 + 2y^2\right)\Bigg|_0^1$$

$$= \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \pi\left(\frac{3 - 25 + 30}{15}\right)$$

$$= \frac{8\pi}{15}$$

Exercício 7. Cada integral representa o volume de um sólido. Descreva o sólido.

(a) $\pi \int_0^{\pi} \operatorname{sen} x \, dx$ (b) $\pi \int_0^{1} y^4 - y^8 \, dy$

(c) $\pi \int_0^{\pi/2} [(1+\cos x)^2 - 1] \, dx$ (d) $\pi \int_1^{4} [3^2 - (3-\sqrt{x})^2] \, dx$

(a) $V = \pi \int_0^{\pi} (\sqrt{\operatorname{sen} x})^2 dx$

Sólido de revolução em torno do eixo x da região entre $y = \sqrt{\operatorname{sen} x}$, $y = 0$, $0 \leq x \leq \pi$.

(b) $V = \pi \int_0^{1} (y^2)^2 - (y^4)^2 dy$

Sólido de revolução em torno do eixo y da região entre $x = y^2$, $x = y^4$, $y \geq 0$.

(c) $V = \pi \int_0^{\pi/2} (1 + \cos x)^2 - 1 \, dx$

Sólido de revolução em torno do eixo x da região entre $y = 1 + \cos x$, $y = 1$, $0 \leq x \leq \frac{\pi}{2}$.

(d) $V = \pi \int_1^4 \left[3^2 - (3-\sqrt{x})^2\right] dx$

Sólido de revolução em torno do eixo x da região entre $y = 3$, $y = (3-\sqrt{x})$, $1 \leq x \leq 4$

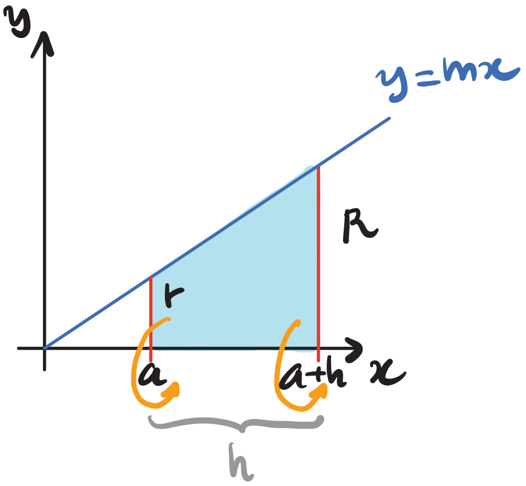
Exercício 8. Encontre os seguintes volumes:

(a) Tronco de cone circular reto com altura h, raio da base inferior R, raio da base superior r.

(b) Calota de esfera de raio r e altura h.

(c) Tronco de pirâmide com base quadrada de lado b, topo quadrado de lado a e altura h.

(a)



O tronco de cone pode ser visto como o sólido de revolução em torno do eixo x da região da figura ao lado, onde

$$\left.\begin{array}{l} am = r \\ (a+h)m = R \end{array}\right\} \Rightarrow \frac{r}{a} = \frac{R}{a+h}$$

$$\Rightarrow ar + hr = aR$$

$$\Rightarrow hr = a(R-r) \Rightarrow a = \frac{hr}{R-r}$$

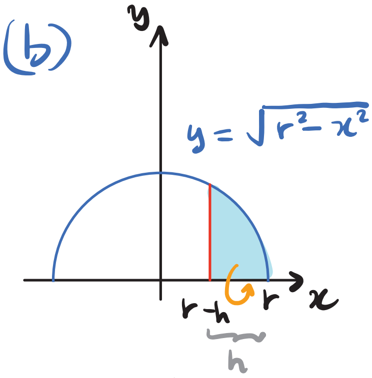
Daí, $m = \dfrac{r}{a} = \dfrac{R-r}{h}$

O volume é

$$V = \pi \int_a^{a+h} (mx)^2 \, dx = \int_{\frac{hr}{R-r}}^{\frac{hR}{R-r}} \pi \left(\frac{R-r}{h}\right)^2 x^2 \, dx$$

$$= \frac{\pi}{3}\left(\frac{R-r}{h}\right)^2 x^3 \Big|_{\frac{hr}{R-r}}^{\frac{hR}{R-r}} = \frac{\pi}{3} \, \frac{h}{R-r} \cdot (R^3 - r^3)$$

(b)



A calota esférica pode ser vista como o sólido de revolução em torno do eixo x da região da figura ao lado.

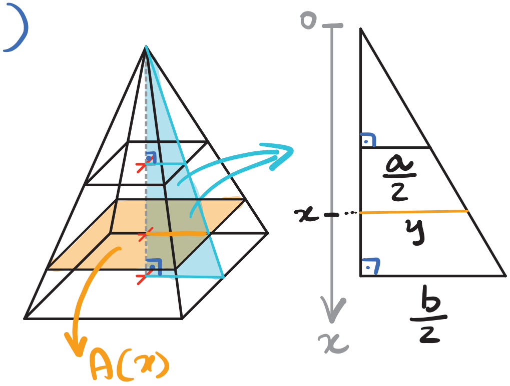
$$V = \pi \int_{r-h}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{r-h}^{r} r^2 - x^2 \, dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{r-h}^{r} = \pi r^2 h - \frac{\pi}{3}\left[r^3 - (r-h)^3\right]$$

$$= \cancel{\pi r^2 h} - \frac{\pi}{3}\left[\cancel{r^3} - \left(\cancel{r^3} - \cancel{3r^2 h} + 3rh^2 - h^3\right)\right]$$

$$= \frac{\pi}{3}\left(3rh^2 - h^3\right) = \frac{\pi}{3} h^2 (3r - h)$$

(c)

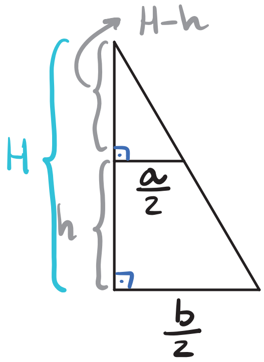


A área $A(x)$ da seção a uma distância x do vértice da pirâmide é

$$A(x) = 4y^2,$$

onde y satisfaz

$$\frac{b/2}{H} = \frac{y}{x} \Rightarrow y = \frac{bx}{2H},$$ H é a altura da pirâmide.



Para encontrar H, temos

$$\frac{H-h}{a/2} = \frac{H}{b/2} \Rightarrow bH - bh = aH$$

$$\Rightarrow H = \frac{bh}{b-a}$$

Assim,

$$A(x) = 4y^2 = 4\left(\frac{bx}{2H}\right)^2 = \left(\frac{\cancel{b}x}{\cancel{b}h}\right)^2 (b-a)^2$$

$$= x^2\left(\frac{b-a}{h}\right)^2$$

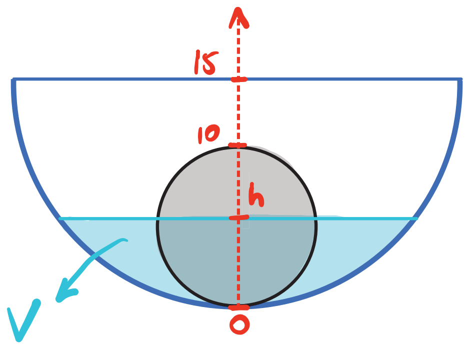
Logo,

$$V = \int_{H-h}^{H} A(x)\, dx = \int_{H-h}^{H} x^2\left(\frac{b-a}{h}\right)^2 dx$$

$$= \frac{1}{3}\left(\frac{b-a}{h}\right)^2 x^3 \Bigg|_{\frac{ah}{b-a}}^{\frac{bh}{b-a}}$$

$$= \frac{1}{3}\,\frac{h}{b-a}\,(b^3 - a^3) = \frac{1}{3}\,\frac{b^3-a^3}{b-a}\cdot h$$

Exercício 9. Uma tigela tem a forma de um hemisfério com diâmetro de 30 cm. Uma bola pesada com diâmetro de 10 cm é colocada dentro da tigela, e depois despeja-se água até uma profundidade de h centímetros. Encontre o volume de água na tigela.



Para $0 < h \leq 10$, o volume de água será a diferença entre os volumes das calotas.

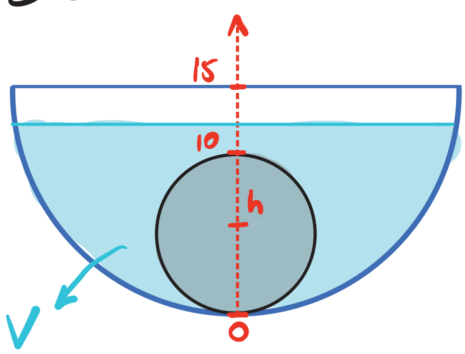
Vimos num exercício anterior que o volume da calota de altura h numa esfera de raio r é

$$V_{r,h} = \frac{\pi}{3} h^2 (3r - h)$$

Assim, se $0 < h \leq 10$, o volume de água é, para $R = 15$, $r = 5$,

$$V = \frac{\pi}{3} h^2 (3R - h) - \frac{\pi}{3} h^2 (3r - h)$$
$$= \pi h^2 (R - r)$$

Se $10 < h \leq 15$, o volume de água é o volume da calota de raio maior R menos o volume da esfera de raio r.



Assim, se $10 < h \leq 15$,

$$V = \frac{\pi}{3} h^2 (3R - h) - \frac{4}{3}\pi r^3$$

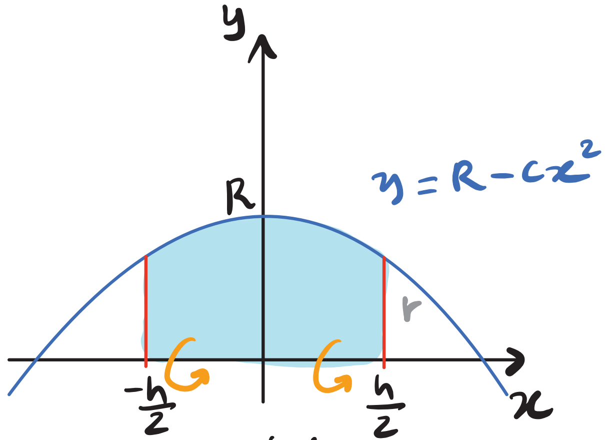
Logo, temos

$$V(h) = \begin{cases} \pi h^2 (R - r) & (0 < h \leq 2r) \\ \frac{\pi}{3} h^2 (3R - h) - \frac{4}{3}\pi r^3 & (2r < h \leq 2R) \end{cases}$$

onde $R = 15$ e $r = 5$.

Exercício 10. Um barril com altura h e raio máximo R é construído pela rotação em torno do eixo x da parábola $y = R - cx^2$, $-h/2 \leqslant x \leqslant h/2$, onde c é uma constante positiva. Mostre que o raio de cada extremidade do barril é $r = R - d$, onde $d = ch^2/4$. Em seguida, mostre que o volume delimitado pelo barril é

$$V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$$



Se r é o raio da extremidade, então

$$r = y\left(\frac{h}{2}\right) = R - \underbrace{\frac{ch^2}{4}}_{d}.$$

O volume é

$$V = \pi\int_{-h/2}^{h/2}(R - cx^2)^2\,dx = \pi\int_{-h/2}^{h/2} R^2 - 2Rcx^2 + c^2x^4\,dx$$

$$= \pi\left(R^2x - \frac{2Rc}{3}x^3 + \frac{c^2}{5}x^5\right)\Bigg|_{h/2}^{h/2}$$

$$= \cancel{2}\pi\left(R^2\frac{h}{\cancel{2}} - \frac{2Rch^3}{3\,\cancel{2}\cdot 4} + \frac{c^2h^5}{5\cdot\cancel{2}\cdot 4^2}\right) \qquad \left(d = \frac{ch^2}{4}\right)$$

$$= \pi h\left(R^2 - \frac{2}{3}Rd + \frac{d^2}{5}\right)$$

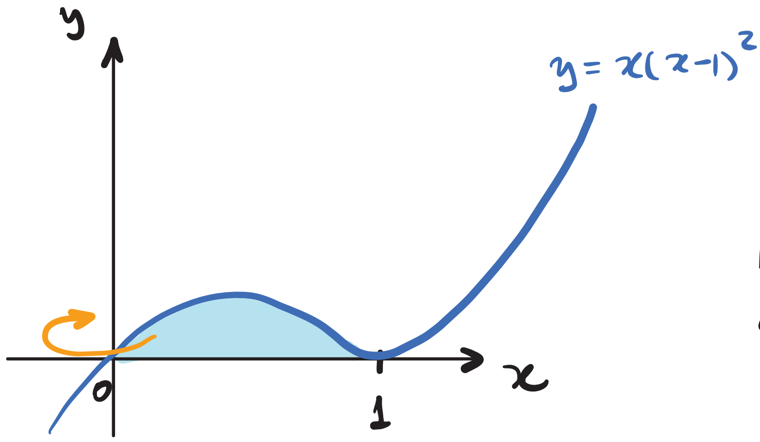
$$r = R - d \Rightarrow r^2 = R^2 - 2Rd + d^2$$

$$\Rightarrow -\frac{2}{3}Rd = \frac{r^2 - R^2 - d^2}{3}$$

$$= \pi h\left(\frac{3R^2}{3} + \frac{r^2}{3} - \frac{R^2}{3} - \frac{d^2}{3} + \frac{d^2}{5}\right)$$

$$= \frac{\pi h}{3}\left(2R^2 + r^2 - \frac{2d^2}{5}\right)$$

Exercício 11. Calcule o volume do sólido obtido pela rotação da região entre as curvas $y = 0$ e $y = x(x-1)^2$ em torno do eixo y.



Esse é um exemplo onde é mais calcular pelo método das cascas cilíndricas.

Assim,

$$V = \int_0^1 2\pi x\, f(x)\, dx = \int_0^1 2\pi x^2 (x^2 - 2x + 1)\, dx$$

$$= \int_0^1 2\pi (x^4 - 2x^3 + x^2)\, dx$$

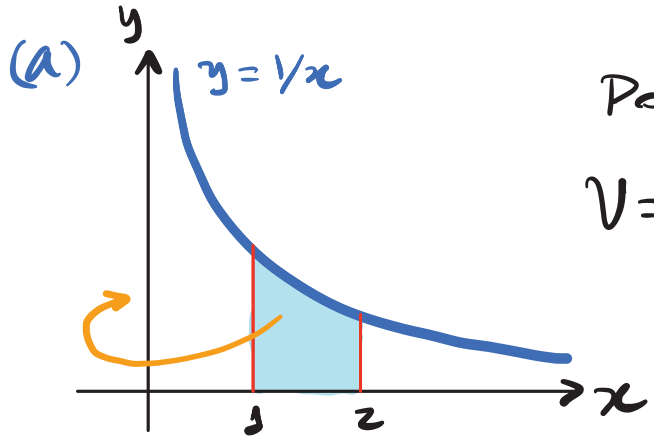
$$= 2\pi \left(\frac{x^5}{5} - \frac{2}{4} x^4 + \frac{x^3}{3} \right) \Big|_0^1$$

$$= 2\pi \left(\frac{1}{5} - \frac{1}{4} + \frac{1}{3} \right) = 2\pi \left(\frac{12 - 15 + 20}{60} \right)$$

$$= \frac{17\pi}{30}$$

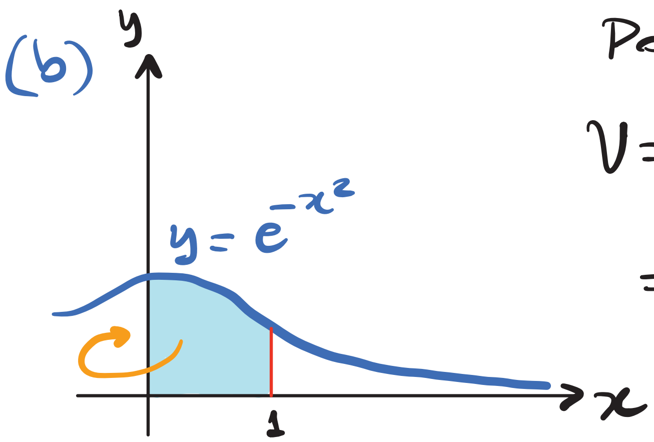
Exercício 12. Calcule os volumes dos sólidos de revolução em torno do eixo y das seguintes regiões:

(a) $y = 1/x$, $y = 0$, $x = 1$, $x = 2$ (b) $y = e^{-x^2}$, $y = 0$, $x = 0$, $x = 1$
(c) $y = x^2$, $y = 6x - 2x^2$



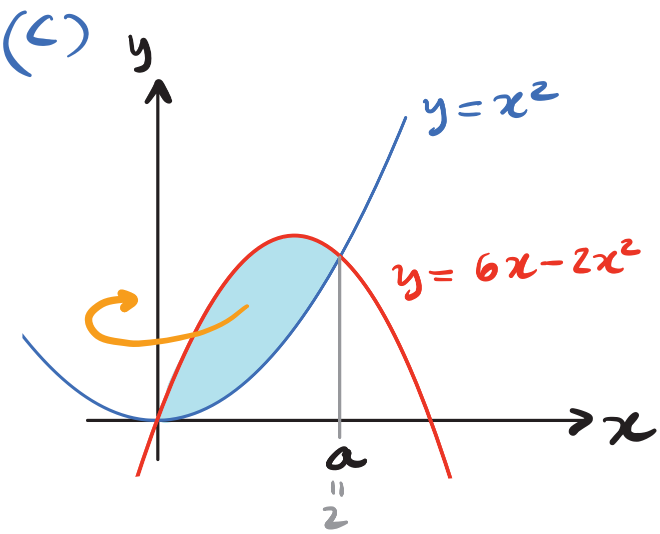
Por cascas cilíndricas:

$$V = \int_1^2 2\pi x \cdot \frac{1}{x}\, dx = 2\pi$$



Por cascas cilíndricas:

$$V = \int_0^1 2\pi x e^{-x^2} dx$$
$$= -\pi e^{-x^2}\Big|_0^1$$
$$= \pi(1 - e^{-1})$$



A interseção em $a > 0$ satisfaz

$$a^2 = 6a - 2a^2$$
$$\Rightarrow 3a = 6 \Rightarrow a = 2$$

Por cascas cilíndricas,

$$V = \int_0^2 2\pi x\left[(6x - 2x^2) - x^2\right] dx$$

$$= 2\pi \int_0^2 6x^2 - 3x^3 \, dx$$

$$= 6\pi \left(x^3 - \frac{1}{4}x^4\right)_0^2 = 6\pi\,(8-4)$$

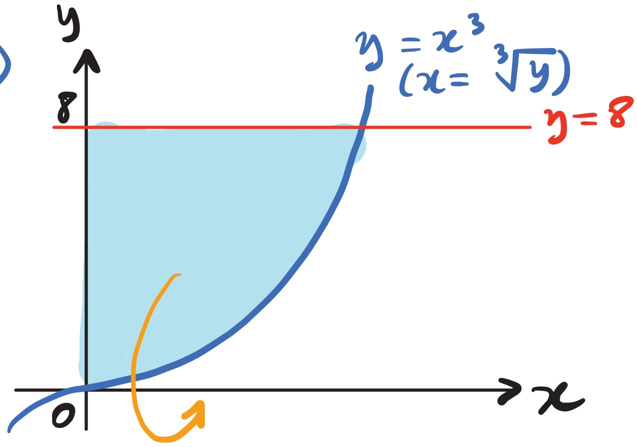
$$= 24\pi$$

Exercício 13. Calcule os volumes dos sólidos de revolução em torno do eixo x das seguintes regiões:

(a) $y = x^3$, $y = 8$, $x = 0$ (b) $y = x^{3/2}$, $y = 8$, $x = 0$

(c) $x = 1 + (y-2)^2$, $x = 2$ (d) $x + y = 4$, $x = y^2 - 4y + 4$

(a)



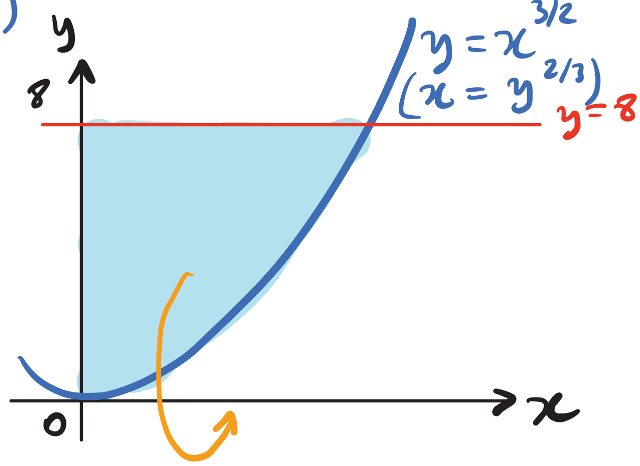
Fazendo $x = \sqrt[3]{y}$ e usando cascas cilíndricas,

$$V = 2\pi \int_0^8 y \sqrt[3]{y}\, dy$$

$$= 2\pi \int_0^8 y^{4/3}\, dy = 2\pi \cdot \frac{3}{7} y^{7/3} \Big|_0^8$$

$$= \frac{6\pi}{7} \cdot 2^7 = \frac{3\pi}{7} \cdot 2^8$$

(b)



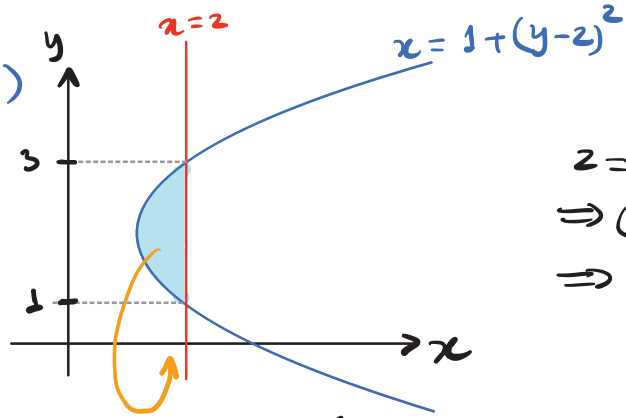
$$x = y^{2/3}$$

$$V = 2\pi \int_0^8 y \cdot y^{2/3}\, dy$$

$$= 2\pi \cdot \frac{3}{5} y^{5/3} \Big|_0^8$$

$$= \frac{3\pi}{5} \cdot 2^6$$

(c)



Interseções:

$$2 = 1 + (y-2)^2$$

$$\Rightarrow (y-2)^2 = 1$$

$$\Rightarrow y = 2 \pm 1 = 1 \text{ e } 3$$

Por cascas cilíndricas, o volume é

$$V = 2\pi \int_1^3 y\left[2 - 1 - (y-2)^2\right] dy$$

$$= 2\pi \int_1^3 y - y^3 + 4y^2 - 4y \, dy$$

$$= 2\pi \int_1^3 -y^3 + 4y^2 - 3y \, dy$$

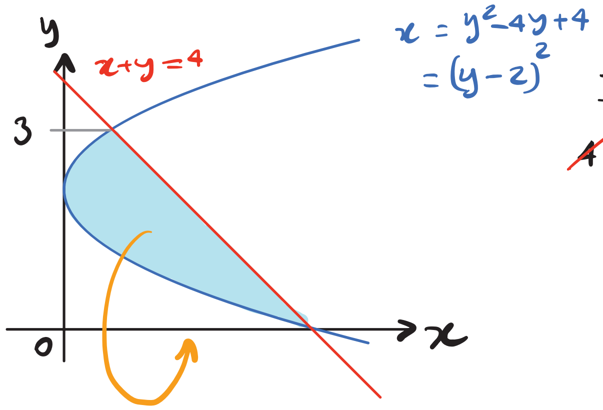
$$= 2\pi \left(-\frac{y^4}{4} + \frac{4y^3}{3} - \frac{3y^2}{2} \right) \Bigg|_1^3$$

$$= 2\pi \cdot 9 \left(-\frac{9}{4} + \frac{12}{3} - \frac{3}{2} \right) - 2\pi \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2} \right)$$

$$= 18\pi \left(\frac{-27 + 48 - 18}{12} \right) - 2\pi \left(\frac{-3 + 16 - 18}{12} \right)$$

$$= \frac{9\pi}{2} + \frac{5\pi}{6} = \frac{32\pi}{6} = \frac{16\pi}{3}$$

(d)



Intersecções:

$\cancel{4}-y = y^2-4y+\cancel{4}$

$y(y-3)=0$

$\Rightarrow y=0$ e 3

$$V = 2\pi\int_0^3 y\left(\cancel{4}-y-y^2+4y-\cancel{4}\right)dy$$

$$= 2\pi\int_0^3 3y^2-y^3\,dy = 2\pi\left(y^3-\frac{y^4}{4}\right)\Big|_0^3$$

$$= 2\pi\left(27-\frac{81}{4}\right) = \frac{27\pi}{2}$$

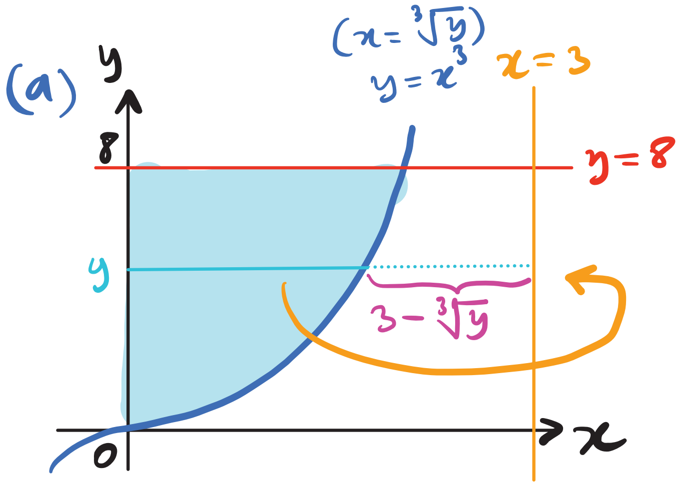
Exercício 14. Calcule o volume gerado pela rotação da região delimitada pelas curvas dadas em torno da reta dada:

(a) $y = x^3$, $y = 8$, $x = 0$ em torno de $x = 3$

(b) $y = x^2$, $y = 2 - x^2$ em torno de $x = 1$

(c) $y = \sqrt{x}$, $x = 2y$ em torno de $x = 5$

(a)



O anel da seção em y tem área

$$A(y) = \pi\left[3^2 - (3 - \sqrt[3]{y})^2\right]$$

$$= \pi\left(6y^{1/3} - y^{2/3}\right)$$

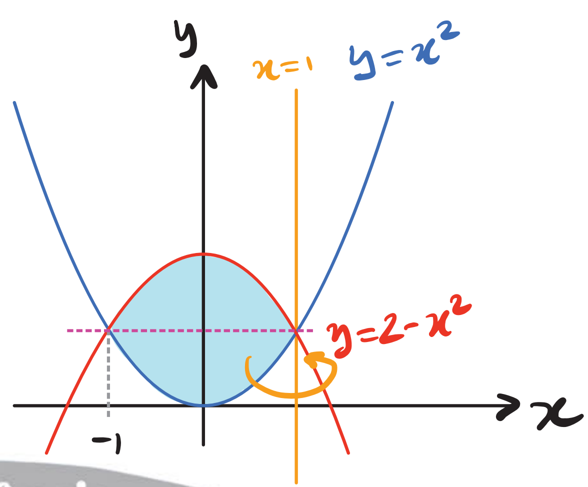
Logo

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi\left(6y^{1/3} - y^{2/3}\right)dy$$

$$= \pi\left(\frac{9}{2}y^{4/3} - \frac{3}{5}y^{5/3}\right)\Bigg|_0^8$$

$$= \pi\left(72 - \frac{96}{5}\right) = \frac{264\pi}{5}$$

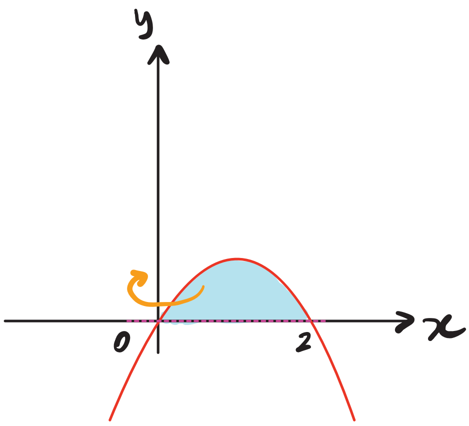
(b)



Interseção:

$$x^2 = 2 - x^2 \Rightarrow 2x^2 = 2$$
$$\Rightarrow x = \pm 1$$

Existe uma simetria: o volume acima da linha rosa é igual ao volume abaixo. Esse volume é equivalente ao da rotação da região entre $y = 1 - (x-1)^2 = 2x - x^2$ e o eixo x ao redor do eixo y:



Assim, o volume procurado é

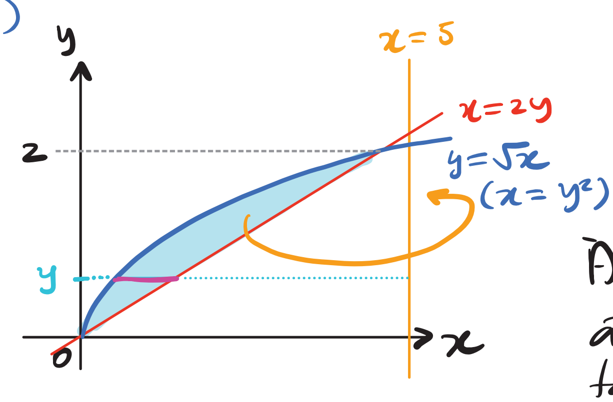
$$V = 2 \cdot \int_0^2 2\pi x (2x - x^2)\, dx$$

$$= 4\pi \int_0^2 2x^2 - x^3\, dx$$

$$= 4\pi \left(\frac{2}{3}x^3 - \frac{x^4}{4} \right)\Big|_0^2$$

$$= 4\pi \left(\frac{16}{3} - \frac{16}{4} \right) = \frac{64\pi}{12} = \frac{16\pi}{3}$$

(c)



Interseção $(y > 0)$:

$$2y = y^2$$
$$\Rightarrow y = 2$$

À altura y, a área do anel resultante da seção do sólido é

$$A(y) = \pi \left[(5 - y^2)^2 - (5 - 2y)^2 \right]$$

$$= \pi\left[\, y^4 - 10y^2 - 4y^2 + 20y \,\right]$$

$$= \pi\left(y^4 - 14y^2 + 20y\right)$$

Logo,

$$V = \int_0^2 A(y)\,dy = \int_0^2 \pi\left(y^4 - 14y^2 + 20y\right)\,dy$$

$$= \pi\left(\frac{y^5}{5} - \frac{14y^3}{3} + \frac{\cancelto{10}{20}\,y^2}{\cancelto{1}{2}}\right)\Bigg|_0^2$$

$$= 4\pi\left(\frac{8}{5} - \frac{28}{3} + 10\right)$$

$$= \frac{4\pi}{15}\cdot\left(24 - 140 + 150\right) = \frac{136\pi}{15}$$

Exercício 1. Calcule as integrais usando integração por partes com as escolhas de u e dv indicadas:

(a) $\int xe^{2x}\,dx$, $\quad u = x$, $\quad dv = e^{2x}\,dx$ $\qquad$ (b) $\int \sqrt{x}\log x\,dx$, $\quad u = \log x$, $\quad dv = \sqrt{x}\,dx$.

(a)
$$\int \underbrace{x}_{u}\underbrace{e^{2x}dx}_{dv} = x\underbrace{\left(\frac{1}{2}e^{2x}\right)}_{v} - \frac{1}{2}\int e^{2x}dx$$
$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$
$$= \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x}$$

(b)
$$\int \sqrt{x}\log x\,dx = \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int \frac{x^{3/2}}{x}dx$$
$$= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int x^{1/2}dx$$
$$= \frac{2}{3}x^{3/2}\log x - \left(\frac{2}{3}\right)^2 x^{3/2}$$

Exercício 2. Calcule a integral

(a) $\int x\cos 5x\,dx$ (b) $\int re^{r/2}\,dr$ (c) $\int (x^2+2x)\cos x\,dx$

(d) $\int \arccos x\,dx$ (e) $\int t^4\log t\,dt$ (f) $\int t\,\text{cossec}^2\,t\,dt$

(g) $\int \log\sqrt[3]{x}\,dx$ (h) $\int t\,\text{sen}\,2t\,dt$ (i) $\int \arctan 2y\,dy$

(j) $\int x\cosh ax\,dx$ (k) $\int \frac{z}{10^z}\,dz$ (l) $\int \arctan 4t\,dt$

(m) $\int z^3e^z\,dz$ (n) $\int \frac{xe^{2x}}{(1+2x)^2}\,dx$ (o) $\int \frac{\log x}{x^2}\,dx$

(p) $\int_0^{\pi} x\,\text{sen}\,x\cos x\,dx$ (q) $\int_1^5 \frac{x}{e^x}\,dx$ (r) $\int_0^{\pi/3} \text{sen}\,x\log(\cos x)\,dx$

(s) $\int_4^9 \frac{\log y}{\sqrt{y}}\,dy$ (t) $\int x\tan^2 x\,dx$ (u) $\int (\text{arcsen}\,x)^2\,dx$

(v) $\int_1^2 w^2\log w\,dw$ (w) $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$ (x) $\int_0^1 \frac{x^3}{\sqrt{4+x^2}}\,dx$

(y) $\int_0^t e^s\,\text{sen}(t-s)\,ds$ (z) $\int \arctan\sqrt{x}\,dx$

(a) $\int x\cos 5x\,dx = \frac{1}{5}x\,\text{sen}\,5x - \int \text{sen}\,5x\,dx$

$= \frac{x}{5}\,\text{sen}\,5x + \frac{1}{5}\cos 5x$

(b) $\int re^{r/2}\,dr = 2re^{r/2} - 2\int e^{r/2}\,dr$

$=(2r-4)e^{r/2}$

(c) $\int (x^2+2x)\cos x\,dx = (x^2+2x)\,\text{sen}\,x$

$-2\int (x+1)\,\text{sen}\,x\,dx = (x^2+2x)\,\text{sen}\,x$

$+2(x+1)\cos x - 2\int \cos x\,dx$

$=(x^2+2x-2)\,\text{sen}\,x + 2(x+1)\cos x$

(d) $\int \arccos x \, dx = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx$

$= x \arccos x - \sqrt{1-x^2}$

(e) $\int t^4 \log t \, dt = \frac{t^5}{5} \log t - \frac{1}{5} \int \frac{\cancel{t^5}^{4}}{\cancel{t}} \, dt$

$= \frac{t^5}{5} \log t - \frac{1}{5^2} t^5$

(f) $\int t \operatorname{cossec}^2 t \, dt = -t \cot t + \int \frac{\cos t}{\operatorname{sen} t} \, dt$

$= \log |\operatorname{sen} t| - t \cdot \cot t$

(g) $\int \log \sqrt[3]{x} \, dx = \frac{1}{3} \int \log x \, dx$

$= \frac{1}{3} x \log x - \frac{1}{3} \int \frac{\cancel{x}}{\cancel{x}} \, dx$

$= \frac{1}{3} (x \log x - x)$

(h) $\int t \operatorname{sen} 2t \, dt = \frac{-1}{2} t \cos 2t + \frac{1}{2} \int \cos 2t \, dt$

$= \frac{\operatorname{sen} 2t}{4} - \frac{t}{2} \cos 2t$

(i) $\int \arctan 2y \, dy = y \arctan 2y - \int \frac{y}{1+4y^2} dy = y \arctan 2y - \frac{1}{8} \log(1+4y^2)$

(j) $\int x \cosh ax \, dx = \frac{1}{a} x \,\text{senh}\, ax - \frac{1}{a} \int \text{senh}\, ax \, dx$

$= \frac{x}{a} \,\text{senh}\, ax - \frac{1}{a^2} \cosh ax$

(k) $\int z \cdot 10^{-z} dz = \frac{-1}{\log 10} z\, 10^{-z} + \frac{1}{\log 10} \int 10^{-z} dz$

$= -\frac{z \cdot 10^{-z}}{\log 10} - \frac{1}{(\log 10)^2} \cdot 10^{-z}$

(l) $\int \arctan 4t \, dt = t \arctan 4t - \int \frac{t}{1+16t^2} dt$

$= t \cdot \arctan 4t - \frac{1}{32} \log(1+16t^2)$

(m) $\int z^3 e^z dz = z^3 e^z - \int 3z^2 e^z dz$

$= z^3 e^z - 3z^2 e^z + \int 6z e^z dz$

$= (z^3 - 3z^2 + 6z) e^z - 6 \int e^z dz$

$= (z^3 - 3z^2 + 6z - 6) e^z$

(n) $\displaystyle\int \frac{xe^{2x}}{(1+2x)^2}dx = -\frac{1}{2}\frac{xe^{2x}}{1+2x} + \frac{1}{2}\int \frac{\cancel{(1+2x)}e^{2x}}{\cancel{1+2x}}dx$

$$= \left(\frac{1}{4} - \frac{1}{2}\frac{x}{1+2x}\right)e^{2x}$$

$$= \frac{1+2x-2x}{4+8x}e^{2x} = \frac{e^{2x}}{8x+4}$$

(o) $\displaystyle\int \frac{\log x}{x^2}dx = -\frac{1}{x}\log x + \int \frac{dx}{x^2}$

$$= -\frac{\log x}{x} - \frac{1}{x}$$

(p) $\displaystyle\int_0^{\pi} x\underbrace{\operatorname{sen}x\cos x}_{\frac{1}{2}\operatorname{sen}2x}dx = \int_0^{\pi}\frac{x}{2}\operatorname{sen}2x\,dx$

$$= -\frac{x}{4}\cos 2x\Big|_0^{\pi} + \frac{1}{4}\int_0^{\pi}\cos 2x\,dx$$

$$= -\frac{1}{4} + \frac{1}{8}\operatorname{sen}2x\Big|_0^{\pi} = -\frac{1}{4}$$

(q) $\displaystyle\int_1^5 xe^{-x}dx = -xe^{-x}\Big|_1^5 + \int_1^5 e^{-x}dx$

$$= -5e^{-5} + e^{-1} - e^{-x}\Big|_1^5$$

$$= -5e^{-5} + e^{-1} - e^{-5} + e^{-1}$$

$$= 2e^{-1} - 6e^{-5}$$

(r) $\displaystyle\int_0^{\pi/3} \operatorname{sen} x \log(\cos x)\, dx = -\cos x \log(\cos x)\Big|_0^{\pi/3}$

$\displaystyle - \int_0^{\pi/3} \frac{\cancel{\cos x}}{\cancel{\cos x}} \cdot \operatorname{sen} x \, dx = -\frac{1}{2}\log\frac{1}{2} + \cos x \Big|_0^{\pi/3}$

$= \frac{1}{2} + \frac{1}{2}\log 2 - 1$

$= \frac{1}{2}(\log 2 - 1)$

(s) $\displaystyle\int_4^9 \frac{\log y}{\sqrt{y}}\, dy = 2\sqrt{y}\log y\Big|_4^9 - 2\int_4^9 \frac{\cancel{\sqrt{y}}}{\cancel{y}\,\sqrt{y}}\, dy$

$= \left(2\sqrt{y}\log y - 4\sqrt{y}\right)\Big|_4^9$

$= 12\log 3 - 12 - 8\log 2 + 8$

$= 4\log\frac{9}{4} - 4$

(t) $\displaystyle\int x\tan^2 x\, dx = \int x(\sec^2 x - 1)\, dx$

$\displaystyle = x\tan x - \int \tan x\, dx - \frac{x^2}{2}$

$\displaystyle = x\tan x - \frac{x^2}{2} + \int -\frac{\operatorname{sen} x}{\cos x}\, dx$

$= x\tan x - \frac{x^2}{2} + \log|\cos x|$

(u) $\displaystyle\int (\operatorname{arcsen} x)^2\,dx = x(\operatorname{arcsen} x)^2 - 2\int \frac{x \operatorname{arcsen} x}{\sqrt{1-x^2}}\,dx = x(\operatorname{arcsen} x)^2 + 2\sqrt{1-x^2}\operatorname{arcsen} x - 2\int \frac{\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}}}\,dx$

$$= x(\operatorname{arcsen} x)^2 + 2\sqrt{1-x^2}\operatorname{arcsen} x - 2x$$

(v) $\displaystyle\int_1^2 w^2 \log w\,dw = \frac{1}{3}w^3 \log w\Big|_1^2 - \frac{1}{3}\int_1^2 \frac{w^{\cancel{3}\,2}}{\cancel{w}}\,dw$

$$= \frac{1}{3}w^3\log w - \frac{1}{9}w^3\Big|_1^2$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} + \frac{1}{9}$$

$$= \frac{8}{3}\log 2 - \frac{7}{9}$$

(w) $\displaystyle\int_1^{\sqrt{3}} \arctan\frac{1}{x}\,dx = x\arctan\frac{1}{x}\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{1+\frac{1}{x^2}}\cdot\frac{1}{x^2}\,dx = x\arctan\frac{1}{x}\Big|_1^{\sqrt{3}}$

$$+ \int_1^{\sqrt{3}} \frac{x}{1+x^2}\,dx = x\arctan\frac{1}{x} + \frac{1}{2}\log(1+x^2)\Big|_1^{\sqrt{3}}$$

$$= \sqrt{3}\,\frac{\pi}{6} - \frac{\pi}{4} + \log 2 - \frac{1}{2}\log 2 = \left(\frac{2\sqrt{3}-3}{12}\right)\pi + \frac{1}{2}\log 2$$

(x) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx$

$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2}$

(y) $\displaystyle\int_0^t e^s \operatorname{sen}(t-s)\,ds = e^s \operatorname{sen}(t-s)\Big|_0^t$

$\displaystyle + \int_0^t e^s \cos(t-s)\,ds = -\operatorname{sen} t + e^s\cos(t-s)\Big|_0^t$

$\displaystyle - \int_0^t e^s \operatorname{sen}(t-s)\,dt$

$\displaystyle \Rightarrow 2\int_0^t e^s \operatorname{sen}(t-s)\,dt = -\operatorname{sen} t + e^t - \cos t$

$\displaystyle \Rightarrow \int_0^t e^s \operatorname{sen}(t-s)\,dt = \frac{e^t - \operatorname{sen} t - \cos t}{2}$

(z) $\displaystyle\int \arctan\sqrt{x}\,dx = x\arctan\sqrt{x}$

$\displaystyle - \int \frac{x}{2\sqrt{x}}\cdot\frac{1}{1+x}\,dx = x\arctan\sqrt{x}$

$\displaystyle - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}\,dx$

Vamos calcular a última integral

$u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\,du$

$$\int \frac{\sqrt{x}}{1+x}\,dx = \int \frac{2u^2}{1+u^2}\,du = 2\int \frac{u^2+1-1}{u^2+1}\,du$$

$$= 2u - 2\arctan u$$

Logo,

$$\int \arctan\sqrt{x}\,dx = x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x}$$

$$= (x+1)\arctan\sqrt{x} - \sqrt{x}$$

Exercício 3. Calcule:

(a) $\displaystyle\int e^{\sqrt{x}}\,dx$ (b) $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta$ (c) $\displaystyle\int x\log(1+x)\,dx$

(d) $\displaystyle\int \cos(\log x)\,dx$ (e) $\displaystyle\int_0^{\pi} e^{\cos t}\,\mathrm{sen}\,2t\,dt$ (f) $\displaystyle\int \frac{\mathrm{arcsen}(\log x)}{x}\,dx$

(a) $u=\sqrt{x} \Rightarrow x=u^2 \Rightarrow dx = 2u\,du$

$$\int e^{\sqrt{x}}dx = \int 2u e^{u}du = 2ue^{u} - 2\int e^{u}du$$

$$= 2(u-1)e^{u} = 2(\sqrt{x}-1)e^{\sqrt{x}}$$

(b) $u=\theta^2 \Rightarrow \theta=\sqrt{u} \Rightarrow d\theta = \frac{1}{2\sqrt{u}}du$

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos\theta^2 d\theta = \int_{\pi/2}^{\pi} u^{3/2}\cdot 2u^{-1/2}\cdot \cos u\,du$$

$$= \int_{\pi/2}^{\pi} 2u\cos u\,du = 2u\,\mathrm{sen}\,u\Big|_{\pi/2}^{\pi} - 2\int_{\pi/2}^{\pi}\mathrm{sen}\,u\,du$$

$$= -\pi + 2\cos u\Big|_{\pi/2}^{\pi} = -\pi - 2$$

(c) $u=x+1 \Rightarrow du=dx,\ x=u-1$

$$\int x\log(x+1)\,dx = \int (u-1)\log u\,du$$

$$= \left(\frac{1}{2}u^2-u\right)\log u - \int \frac{\left(\frac{1}{2}u^2-u\right)}{u}du$$

$$= \left(\frac{1}{2}u^2-u\right)\log u - \int\left(\frac{1}{2}u-1\right)du$$

$$= \left(\frac{1}{2}u^2 - u\right)\log u - \frac{u^2}{4} + u$$

$$= \left[\frac{(x+1)^2}{2} - (x+1)\right]\log(x+1) - \frac{(x+1)^2}{4} + (x+1)$$

$$= \frac{(x+1)}{2}(x+1-2)\log(x+1) - \frac{(x+1)}{4}.\left[(x+1) - 4\right]$$

$$= \frac{(x+1)(x-1)}{2}\log(x+1) - \frac{(x+1)(x-3)}{4}$$

(d) $u = \log x \Rightarrow x = e^u \Rightarrow dx = e^u du$

$$\int \cos(\log x)\, dx = \int e^u \cos u\, du$$

$$= e^u \cos u + \int e^u \operatorname{sen} u\, du = e^u(\cos u + \operatorname{sen} u)$$

$$- \int e^u \operatorname{sen} u\, du$$

$$\Rightarrow \int e^u \operatorname{sen} u\, du = \frac{e^u}{2}(\cos u + \operatorname{sen} u)$$

$$\Rightarrow \int \cos(\log x)\, dx = \frac{x}{2}\left[\cos(\log x) + \operatorname{sen}(\log x)\right]$$

(e) $\displaystyle\int_0^\pi e^{\cos t} \operatorname{sen} 2t\, dt = 2\int_0^\pi e^{\cos t} \operatorname{sen} t \cos t\, dt$

$u = \cos t \Rightarrow du = -\operatorname{sen} t\, dt$

$$= 2\int_{-1}^{1} e^u u\, du = 2\cdot\left[u e^u\Big|_{-1}^{1} - \int_{-1}^{1} e^u dt\right]$$

$$= 2(u-1)e^u\Big|_{-1}^{1} = -4e^{-1}$$

(f) $u = \log x \Rightarrow du = \dfrac{dx}{x}$

$$\int \frac{\operatorname{arcsen}(\log x)}{x}\,dx = \int \operatorname{arcsen} u \, du$$

$$= u \operatorname{arcsen} u - \int \frac{u}{\sqrt{1-u^2}}\,du$$

$$= u \operatorname{arcsen} u + \sqrt{1-u^2}$$

$$= \log x \cdot \operatorname{arcsen}(\log x) + \sqrt{1-(\log x)^2}$$

Exercício 4. Encontre:

(a) $\int \operatorname{sen}^2 x\, dx$ (b) $\int \operatorname{sen}^4 x\, dx$

(a) $\int \operatorname{sen}^2 x\, dx = -\operatorname{sen} x \cos x + \int \cos^2 x\, dx$

$= -\operatorname{sen} x \cos x + \int 1 - \operatorname{sen}^2 x\, dx$

$= -\frac{\operatorname{sen} 2x}{2} + x - \int \operatorname{sen}^2 x\, dx$

$\Rightarrow \int \operatorname{sen}^2 x\, dx = \frac{x}{2} - \frac{1}{4} \operatorname{sen} 2x$

(b) $\int \operatorname{sen}^4 x\, dx = -\cos x \operatorname{sen}^3 x + 3\int \cos^2 x \operatorname{sen}^2 x\, dx$

$= -\cos x \operatorname{sen}^3 x + 3\int \operatorname{sen}^2 x\, dx - 3\int \operatorname{sen}^4 x\, dx$

$= -\cos x \operatorname{sen}^3 x + \frac{3}{2} x - \frac{3}{4} \operatorname{sen} 2x$

$- 3\int \operatorname{sen}^4 x\, dx$

$\Rightarrow \int \operatorname{sen}^4 x\, dx = -\frac{1}{4} \cos x \operatorname{sen}^3 x + \frac{3}{4} x - \frac{3}{16} \operatorname{sen} 2x$

Exercício 5. Mostre as seguintes fórmulas de redução:

(a) $\displaystyle\int (\log x)^n\,dx = x(\log x)^n - n\int (\log x)^{n-1}\,dx$

(b) $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\mathrm{sen}\,x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

(c) $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

(d) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(e) $\displaystyle\int \sec^n x\,dx = \frac{\tan x\sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(a) $$\int \underbrace{(\log x)^n}_{u}\,\underbrace{dx}_{} = \underbrace{x}_{v}(\log x)^n - n\int x(\log x)^{n-1}\cdot\frac{1}{x}\,dx$$
$$= x(\log x)^n - n\int(\log x)^{n-1}\,dx$$

(b) $$\int \cos^n x\,dx = \int \cos^{n-1}x\cdot\cos x\,dx = \cos^{n-1}x\,\mathrm{sen}\,x$$
$$+ (n-1)\int \cos^{n-2}x\cdot\mathrm{sen}^2 x\,dx = \cos^{n-1}x\,\mathrm{sen}\,x$$
$$+ (n-1)\int\cos^{n-2}x\,dx - (n-1)\int\cos^n x\,dx$$
$$\Rightarrow n\int\cos^n x\,dx = \cos^{n-1}x\,\mathrm{sen}\,x + (n-1)\int\cos^{n-2}x\,dx$$
$$\Rightarrow \int\cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\,\mathrm{sen}\,x + \frac{n-1}{n}\int\cos^{n-2}x\,dx$$

(c) $$\int x^n \underbrace{e^x dx}_{dv} = x^n e^x - n\int x^{n-1}e^x\,dx$$

(d) $\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx$

$= \int \tan^{n-2} x \cdot (\sec^2 - 1)\,dx$

$= \int \tan^{n-2} x \cdot \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= \tan^{n-1} x - (n-2)\int \tan^{n-3} x \cdot \sec^2 x \tan x\,dx$

$- \int \tan^{n-2} x\,dx$

$= \tan^{n-1} x - (n-2)\int \tan^{n-2}(1 + \tan^2 x)\,dx$

$- \int \tan^{n-2} x\,dx$

$= \tan^{n-1} x - (n-1)\int \tan^{n-2} dx$

$-(n-2)\int \tan^n x\,dx$

$\Rightarrow (n-1)\int \tan^n x\,dx = \tan^{n-1} x - (n-1)\int \tan^{n-2} x\,dx$

$\Rightarrow \int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(e) $\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$$= \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^{n-3}x \cdot \sec x \tan x \cdot \tan x \, dx$$

$$= \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^{n-2}x \cdot (\sec^2 x - 1)\, dx$$

$$= \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^{n}x \, dx + (n-2)\int \sec^{n-2}x \, dx$$

$$\Rightarrow (n-1)\int \sec^{n}x \, dx = \sec^{n-2}x \cdot \tan x + (n-2)\int \sec^{n-2}x \, dx$$

$$\Rightarrow \int \sec^{n}x \, dx = \frac{\sec^{n-2}x \cdot \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x \, dx$$

Exercício 6. Mostre que se $n \in \mathbb{N}$ então $\int_0^1 (\log x)^n \, dx = (-1)^n n!$

Temos que

$$I_n = \int_0^1 (\log x)^n dx = x(\log x)^n \Big|_0^1 - n\int_0^1 \cancel{x}(\log x)^{n-1} \cdot \frac{1}{\cancel{x}} dx$$

$$= -n\int_0^1 (\log x)^{n-1} dx$$

Ou seja,

$$I_n = -n \cdot I_{n-1}$$

Portanto, a aplicação repetida dessa identidade resulta em

$$I_n = (-1)^n n! \, I_0$$
$$= (-1)^n n! \cdot \int_0^1 dx$$
$$= (-1)^n \cdot n!$$

Exercício 7. Mostre que se $n \in \mathbb{N}$ então $\int_0^1 (1-x^2)^n \, dx = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$

$$\int_0^1 (1-x^2)^n dx = x(1-x^2)^n \Big|_0^1 + 2n\int_0^1 x^2(1-x^2)^{n-1} dx$$

$$= 2n \int x^2 (1-x^2)^{n-1} dx$$

$$= \frac{2n}{3} x^3 (1-x^2)^{n-1}\Big|_0^1 + \frac{2^2 n(n-1)}{3}\int_0^1 x^4 (1-x^2)^{n-2} dx$$

$$= \frac{2^2 n(n-1)}{3}\int_0^1 x^4(1-x^2)^{n-2} dx$$

$$= \frac{2^2 n(n-1)}{3\cdot 5} x^5 (1-x^2)^{n-2}\Big|_0^1 + \frac{2^3 n(n-1)(n-2)}{3\cdot 5}\int_0^1 x^6 (1-x^2)^{n-3} dx$$

$$= \frac{2^3 n(n-1)(n-2)}{3\cdot 5}\int_0^1 x^6(1-x^2)^{n-3} dx$$

Se

$$\int_0^1 (1-x^2)^n dx = \frac{2^k n(n-1)\cdots(n-k+1)}{3\cdot 5 \cdot \cdots \cdot (2k-1)}\int_0^1 x^{2k}(1-x^2)^{n-k} dx,$$

então

$$\int_0^1 (1-x^2)^n dx = \frac{2^k n(n-1)\cdots(n-k+1)}{3\cdot 5\cdot \cdots \cdot (2k-1)(2k+1)} x^{2k+1}(1-x^2)^{n-k}\Big|_0^1$$

$$+ \frac{2^k n(n-1)\cdots(n-k+1)(n-k)}{3\cdot 5\cdot \cdots \cdot (2k-1)(2k+1)}\int_0^1 x^{2(k+1)}(1-x^2)^{n-(k+1)} dx$$

$$= \frac{2^k n(n-1)\cdots(n-k+1)(n-k)}{3\cdot 5\cdot \cdots \cdot (2k-1)(2k+1)}\int_0^1 x^{2(k+1)}(1-x^2)^{n-(k+1)} dx$$

Assim, repetindo a ideia para $k = 1, 2, 3, \dots, n$, chegamos em

$$\int_0^1 (1-x^2)^n dx = \frac{2^n \cdot n!}{3 \cdot 5 \cdot \dots (2n-1)} \int_0^1 x^{2n} dx$$

$$= \frac{2^n \cdot n!}{3 \cdot 5 \cdots (2n-1)(2n+1)} \quad x^{2n+1} \Big|_0^1$$

$$= \frac{2^n \cdot n!}{3 \cdot 5 \cdots (2n-1)(2n+1)}$$

O denominador é o produto $I_{(n+1)}$ (n+1) primeiros ímpares. Note que

$(2n+1)! = I_{n+1} \cdot 2 \cdot 4 \cdot 6 \cdot 8 \dots (2n) = I_{n+1} \cdot P_n$

onde P_n é o produto dos n primeiros pares.

Temos

$$P_n = \prod_{k=1}^{n} (2k) = 2^n \cdot n!$$

$$\Rightarrow I_{n+1} = \frac{(2n+1)!}{2^n \cdot n!}$$

Logo,

$$\int_0^1 (1-x^2)^n dx = \frac{2^n \cdot n!}{I_{n+1}} = \frac{2^{2n} \cdot (n!)^2}{(2n+1)!}$$

Exercício 8. Mostre que, para todo $n \in \mathbb{N}^*$, $\int e^{-x^2} x^{2n+1}\,dx$ pode ser expressa por meio de funções elementares.

$$\int e^{-x^2} x^{2n+1}\,dx = -\frac{1}{2}\int -2x\, e^{-x^2} x^{2n}\,dx$$

$$= -\frac{1}{2} e^{-x^2} x^{2n} + \frac{1}{2}\cdot 2n \int e^{-x^2} x^{2n-1}\,dx$$

Ou seja, se

$$I_n = \int e^{-x^2} x^{2n+1}\,dx$$

então

$$I_n = -\frac{1}{2} x^{2n} e^{-x^2} + n\, I_{n-1}$$

Logo,

$$I_n = \left[-\frac{1}{2} x^{2n} - \frac{n}{2} x^{2(n-1)}\right] e^{-x^2} + n(n-1)\, I_{n-2}$$

$$= -\frac{1}{2}\left[x^{2n} + n\, x^{2(n-1)} + n(n-1)\, x^{2(n-2)}\right] e^{-x^2} + n(n-1)(n-2)\, I_{n-3}$$

$$= \cdots =$$

$$= -\frac{1}{2}\sum_{k=0}^{n-1} \frac{n!}{(n-k)!} x^{2(n-k)} e^{-x^2} + n!\, I_0,$$

onde

$$I_0 = \int e^{-x^2} x\,dx = -\frac{1}{2} e^{-x^2}$$

Assim,

$$\int e^{-x^2} x^{2n+1}\,dx = -\frac{1}{2}\sum_{k=0}^{n} \frac{n!}{(n-k)!}\, x^{2(n-k)} e^{-x^2},$$

que é uma função elementar.

Exercício 9. Prove que

$$\int_0^x \left[\int_0^u f(t)\,dt\right] du = \int_0^x f(u)(x-u)\,du$$

$$\int_0^x \Big(\underbrace{\int_0^u f(t)\,dt}_{F(u)}\Big) du = \int_0^x F(u)\,du = -(x-u)F(u)\Big|_0^x$$

$$+ \int_0^x (x-u)\,F'(u)\,du = \int_0^x (x-u) f(u)\,du,$$

onde na última igualdade usamos o teorema fundamental do cálculo

$$F'(u) = \frac{d}{du}\int_0^u f(t)\,dt = f(u)$$

e o fato de que

$$F(0) = \int_0^0 f(t)\,dt = 0.$$

Exercício 10. Prove que

$$\binom{n}{k} = \left[(n+1)\int_0^1 x^k(1-x)^{n-k}\,dx\right]^{-1}$$

Temos que

$$\int_0^1 (1-x)^n dx = -\frac{(1-x)^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$$

Por outro lado,

$$\int_0^1 (1-x)^n dx = x(1-x)^n\Big|_0^1 + n\int_0^1 x(1-x)^{n-1}dx$$

$$= n\int_0^1 x(1-x)^{n-1}dx$$

$$= \frac{n}{2}x^2(1-x)^{n-1}\Big|_0^1 + \frac{n(n-1)}{2}\int_0^1 x^2(1-x)^{n-2}dx$$

$$= \frac{n(n-1)}{2}\int_0^1 x^2(1-x)^{n-2}dx$$

$$= \frac{n(n-1)(n-2)}{2\cdot 3}x^3(1-x)^{n-2}\Big|_0^1 + \frac{n(n-1)(n-2)}{2\cdot 3}\int_0^1 x^3(1-x)^{n-3}dx$$

$$= \frac{n(n-1)(n-2)}{2\cdot 3}\int_0^1 x^3(1-x)^{n-3}dx$$

$$= \cdots =$$

$$= \frac{n!}{(n-k)!\,k!}\int_0^1 x^k(1-x)^{n-k}dx$$

Logo,

$$\binom{n}{k}\int_0^1 x^k(1-x)^{n-k}\,dx = \frac{1}{n+1}$$

$$\Rightarrow \binom{n}{k} = \left[(n+1)\int_0^1 x^k(1-x)^{n-k}\,dx\right]^{-1}$$

Exercício 1. Calcule:

(a) $\int \operatorname{sen}^3 x \cos^2 x \, dx$ (b) $\int_0^{\pi/2} \operatorname{sen}^7 \theta \cos^5 \theta \, d\theta$ (c) $\int_0^{\pi/2} \cos^2 \theta \, d\theta$

(d) $\int_0^{\pi} \cos^4(2t) \, dt$ (e) $\int_0^{\pi/2} \operatorname{sen}^2 x \cos^2 x \, dx$ (f) $\int \sqrt{\cos\theta} \operatorname{sen}^3 \theta \, d\theta$

(g) $\int \cot x \cos^2 x \, dx$ (h) $\int \operatorname{sen}^2 x \operatorname{sen} 2x \, dx$ (i) $\int t \operatorname{sen}^2 t \, dt$

(j) $\int \tan x \sec^3 x \, dx$ (k) $\int \tan^2 x \, dx$ (l) $\int \tan^4 x \sec^6 x \, dx$

(m) $\int \tan^3 x \sec x \, dx$ (n) $\int \tan^5 x \, dx$ (o) $\int x \sec x \tan x \, dx$

(p) $\int_{\pi/6}^{\pi/4} \cot^2 x \, dx$ (q) $\int \frac{\operatorname{sen}^3 \sqrt{x}}{\sqrt{x}} \, dx$ (r) $\int x \operatorname{sen}^3 x \, dx$

(s) $\int_0^{\pi/2} (2 - \operatorname{sen}\theta)^2 \, d\theta$ (t) $\int \frac{\operatorname{sen}(1/t)}{t^2} \, dt$ (u) $\int \cos\theta \cos^5(\operatorname{sen}\theta) \, d\theta$

(a) $\int \operatorname{sen}^3 x \cos^2 x \, dx = \int (1-\cos^2 x)\cos^2 x \operatorname{sen} x \, dx$

$= \int \cos^2 x \operatorname{sen} x - \cos^4 x \operatorname{sen} x \, dx$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5}$

(b) $\int_0^{\pi/2} \operatorname{sen}^7\theta \cos^5\theta \, d\theta = \int_0^{\pi/2} \operatorname{sen}^7\theta\,(1-\operatorname{sen}^2\theta)^2 \cos\theta \, d\theta$

$= \int_0^{\pi/2} (\operatorname{sen}^7\theta - 2\operatorname{sen}^9\theta + \operatorname{sen}^{11}\theta)\cos\theta \, d\theta$

$= \frac{\operatorname{sen}^8\theta}{8} - \frac{2}{10}\operatorname{sen}^{10}\theta + \frac{\operatorname{sen}^{12}\theta}{12} \Big|_0^{\pi/2}$

$= \frac{1}{8} - \frac{1}{5} + \frac{1}{12} = \frac{15 - 24 + 10}{120} = \frac{1}{120}$

(c) $\displaystyle\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta$

$\displaystyle = \frac{\theta}{2} + \frac{\operatorname{sen} 2\theta}{4} \Big|_0^{\pi/2} = \frac{\pi}{4}$

(d) $\displaystyle\int_0^{\pi} \cos^4 2t \, dt = \int_0^{\pi} \left(\frac{1+\cos 4t}{2}\right)^2 dt$

$\displaystyle = \int_0^{\pi} \frac{1+2\cos 4t+\cos^2 4t}{2} \, dt$

$\displaystyle = \int_0^{\pi} \frac{1}{2} + \cos 4t + \frac{1+\cos 8t}{4} \, dt$

$\displaystyle = \frac{3}{4}t + \frac{\operatorname{sen} 4t}{4} + \frac{\operatorname{sen} 8t}{32} \Big|_0^{\pi} = \frac{3\pi}{4}$

(e) $\displaystyle\int_0^{\pi/2} \operatorname{sen}^2 x \cos^2 x \, dx = \int_0^{\pi/2} \left(\frac{\operatorname{sen} 2x}{2}\right)^2 dx$

$\displaystyle = \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 4x}{2} \, dx$

$\displaystyle = \frac{x}{8} - \frac{\operatorname{sen} 4x}{32} \Big|_0^{\pi/2} = \frac{\pi}{16}$

(f) $\displaystyle\int \sqrt{\cos\theta}\, \operatorname{sen}^3\theta \, d\theta = \int \sqrt{\cos\theta}\,(1-\cos^2\theta)\operatorname{sen}\theta \, d\theta$

$\displaystyle = \int \left(\cos^{1/2}\theta - \cos^{5/2}\theta\right)\operatorname{sen}\theta \, d\theta$

$$= \frac{2}{7}\cos^{7/2}\theta - \frac{2}{3}\cos^{3/2}\theta$$

(g) $$\int \cot x \cos^2 x \, dx = \int \frac{\cos x}{\operatorname{sen} x}(1 - \operatorname{sen}^2 x)\, dx$$

$$= \int \frac{\cos x}{\operatorname{sen} x}\, dx - \int \operatorname{sen} x \cos x \, dx$$

$$= \log|\operatorname{sen} x| - \frac{1}{2}\operatorname{sen}^2 x$$

(h) $$\int \operatorname{sen}^2 x \operatorname{sen} 2x \, dx = \int 2 \operatorname{sen}^3 x \cos x \, dx$$

$$= \frac{1}{2}\operatorname{sen}^4 x$$

(i) $$\int t \operatorname{sen}^2 t \, dt = \int t\left(\frac{1 - \cos 2t}{2}\right) dt$$

$$= \frac{t^2}{4} - \frac{1}{2}\int t \cos 2t \, dt$$

$$= \frac{t^2}{4} - \frac{1}{2}\,\frac{t \operatorname{sen} 2t}{2} + \frac{1}{4}\int \operatorname{sen} 2t \, dt$$

$$= \frac{t^2}{4} - \frac{t \operatorname{sen} 2t}{4} - \frac{\cos 2t}{8}$$

(j) $\int \tan x \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \tan x \, dx$

$= \frac{1}{3} \sec^3 x$

(k) $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$

$= \tan x - x$

(l) $\int \tan^4 x \sec^6 x \, dx$

$= \int \tan^4 x \left(1 + \tan^2 x\right)^2 \cdot \sec^2 x \, dx$

$= \int \left(\tan^4 x + 2\tan^6 x + \tan^8 x\right) \sec^2 x \, dx$

$= \frac{1}{5} \tan^5 x + \frac{2}{7} \tan^7 x + \frac{1}{9} \tan^9 x$

(m) $\int \tan^3 x \sec x \, dx = \int \left(\sec^2 x - 1\right) \tan x \sec x \, dx$

$= \int \sec^2 x \cdot \tan x \sec x \, dx - \int \tan x \sec x \, dx$

$= \frac{1}{3} \sec^3 x - \sec x$

(n) $\displaystyle\int \tan^5 x\,dx = \int \tan x \cdot (\sec^2 x - 1)^2\,dx$

$$= \int \tan x(\sec^4 x - 2\sec^2 x + 1)\,dx$$

$$= \int (\sec^3 x - 2\sec x)\sec x \tan x\,dx + \int \frac{\operatorname{sen} x}{\cos x}\,dx$$

$$= \frac{\sec^4 x}{4} - \sec^2 x - \log|\cos x|$$

(o) $\displaystyle\int x\sec x \tan x\,dx = x\sec^2 x - \int \sec^2 x\,dx$

$$= x\sec^2 x - \tan x$$

(p) $\displaystyle\int_{\pi/6}^{\pi/4} \cot^2 x\,dx = \int_{\pi/6}^{\pi/4} \operatorname{cossec}^2 x - 1\,dx$

$$= -\cot x - x \Big|_{\pi/6}^{\pi/4}$$

$$= \frac{\pi}{6} + \sqrt{3} - 1 - \frac{\pi}{4} = \sqrt{3} - 1 - \frac{\pi}{12}$$

(q) $u = \sqrt{x} \Rightarrow 2\,du = \dfrac{dx}{\sqrt{x}}$

$$\int \frac{\operatorname{sen}^3 \sqrt{x}}{\sqrt{x}}\,dx = \frac{1}{2}\int \operatorname{sen}^3 u\,du$$

$$= \frac{1}{2}\int (1-\cos^2 u)\,\text{sen}\,u\,du$$

$$= \frac{1}{2}\left(\frac{\cos^3 u}{3} - \cos u\right) = \frac{1}{2}\left(\frac{\cos^3\sqrt{x}}{3} - \cos\sqrt{x}\right)$$

(r) $\int x\,\text{sen}^3 x\,dx = \int x\,(1-\cos^2 x)\,\text{sen}\,x\,dx$

$$= x\left(\frac{\cos^3 x}{3} - \cos x\right) - \int \frac{1}{3}\cos^3 x - \cos x\,dx$$

$$= x\left(\frac{1}{3}\cos^3 x - \cos x\right) - \frac{1}{3}\int (1-\text{sen}^2 x)\cos x\,dx$$

$$+ \text{sen}\,x$$

$$= x\left(\frac{1}{3}\cos^3 x - \cos x\right) - \frac{1}{3}\left(\text{sen}\,x - \frac{\text{sen}^3 x}{3}\right)$$

$$+ \text{sen}\,x$$

(s) $\int_0^{\pi/2} (2-\text{sen}\,\theta)^2\,d\theta = \int_0^{\pi/2} 4 - 2\,\text{sen}\,\theta + \text{sen}^2\theta\,d\theta$

$$= \int_0^{\pi/2} 4 - 2\,\text{sen}\,\theta + \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\theta + 2\cos\theta - \frac{\text{sen}\,2\theta}{4}\Big|_0^{\pi/2}$$

$$= \frac{9\pi}{4} - 2$$

(t) $\displaystyle\int \frac{\operatorname{sen}(1/t)}{t^2}\,dt = \cos(1/t)$

(u) $u = \operatorname{sen}\theta \Rightarrow du = \cos\theta\, d\theta$

$$\int \cos^5(\operatorname{sen}\theta)\cos\theta\, d\theta = \int \cos^5 u\, du$$

$$= \int (1-\operatorname{sen}^2 u)^2 \cos u\, du$$

$$= \int (1 - 2\operatorname{sen}^2 u + \operatorname{sen}^4 u)\cos u\, du$$

$$= \frac{1}{5}\operatorname{sen}^5 u - \frac{2}{3}\operatorname{sen}^3 u + \operatorname{sen} u$$

$$= \frac{1}{5}\operatorname{sen}^5(\operatorname{sen}\theta) - \frac{2}{3}\operatorname{sen}^3(\operatorname{sen}\theta) + \operatorname{sen}(\operatorname{sen}\theta)$$

Exercício 2. Calcule

(a) $\displaystyle\int \frac{\operatorname{sen} t}{\cos^3 t}\, dt$ (b) $\displaystyle\int \operatorname{cossec}^4 x \cot^6 x\, dx$ (c) $\displaystyle\int_{\pi/4}^{\pi/2} \operatorname{cossec}^4 \theta \cot^4 \theta\, d\theta$

(d) $\displaystyle\int \operatorname{sen} 8x \cos 5x\, dx$ (e) $\displaystyle\int \operatorname{sen} 2\theta \operatorname{sen} 6\theta\, d\theta$ (f) $\displaystyle\int_0^{\pi/6} \sqrt{1+\cos 2x}\, dx$

(g) $\displaystyle\int \frac{1-\tan^2 x}{\sec^2 x}\, dx$ (h) $\displaystyle\int \frac{dx}{\cos x - 1}$ (i) $\displaystyle\int x \operatorname{sen}^2(x^2)\, dx$

(j) $\displaystyle\int \operatorname{sen} 3x \operatorname{sen} 6x\, dx$ (k) $\displaystyle\int \sec^4(x/2)\, dx$ (l) $\displaystyle\int \frac{\cos x + \operatorname{sen} x}{\operatorname{sen} 2x}\, dx$

(a) $\displaystyle\int \frac{\operatorname{sen} t}{\cos^3 t}\, dt = \frac{1}{2}\,\frac{1}{\cos^2 t}$

(b) $\begin{cases} \cot^2 x + 1 = \operatorname{cossec}^2 x \\ \dfrac{d}{dx}\cot x = -\operatorname{cossec}^2 x \end{cases}$

$$\int \operatorname{cossec}^4 x \cot^6 x\, dx$$

$$= \int (1 - \cot^2 x) \cot^6 x \cdot \operatorname{cossec}^2 x\, dx$$

$$= \int (\cot^8 x - \cot^6 x)(-\operatorname{cossec}^2 x)\, dx$$

$$= \frac{1}{9}\cot^9 x - \frac{1}{7}\cot^7 x$$

(c) $$\int_{\pi/4}^{\pi/2} \operatorname{cossec}^4 \theta \cot^4 \theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1 - \cot^2 \theta) \cot^4 \theta \operatorname{cossec}^2 \theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\cot^6\theta - \cot^4\theta)(-\operatorname{cossec}^2\theta)\, d\theta$$

$$= \frac{1}{7}\cot^7\theta - \frac{1}{5}\cot^5\theta \Big|_{\pi/4}^{\pi/2} = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}$$

(d)

$$\operatorname{sen}[(m+n)x] = \operatorname{sen} mx \cdot \cos nx + \operatorname{sen} nx \cdot \cos mx$$

$$\operatorname{sen}[(m-n)x] = \operatorname{sen} mx \cdot \cos nx - \operatorname{sen} nx \cdot \cos mx$$

$$\Rightarrow \operatorname{sen} mx \cdot \cos nx = \frac{\operatorname{sen}[(m+n)x] + \operatorname{sen}[(m-n)x]}{2}$$

$$\int \operatorname{sen} 8x \cos 5x\, dx = \frac{1}{2}\int \operatorname{sen} 13x + \operatorname{sen} 3x\, dx$$

$$= -\frac{\cos 13x}{26} - \frac{\cos 3x}{6}$$

(e) $\cos[(m+n)x] = \cos mx \cdot \cos nx - \operatorname{sen} mx \cdot \operatorname{sen} nx$

$$\cos[(m-n)x] = \cos mx \cdot \cos nx + \operatorname{sen} mx \cdot \operatorname{sen} nx$$

$$\Rightarrow \operatorname{sen} mx \cdot \operatorname{sen} nx = \frac{\cos[(m-n)x] - \cos[(m+n)x]}{2}$$

$$\int \operatorname{sen} 2\theta \operatorname{sen} 6\theta\, d\theta = \int \frac{\cos(-4\theta) - \cos 8\theta}{2}\, d\theta$$

$$= \int \frac{\cos 4\theta - \cos 8\theta}{2}\, d\theta = \frac{\operatorname{sen} 4\theta}{8} - \frac{\operatorname{sen} 8\theta}{16}$$

(f) $2\cos^2 x - 1 = \cos^2 x - \operatorname{sen}^2 x = \cos 2x$

$\Rightarrow \cos 2x + 1 = 2\cos^2 x$

$$\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2}\cos x\, dx = \sqrt{2}\operatorname{sen} x$$

(g) $$\int \frac{1-\tan^2 x}{\sec^2 x}\, dx = \int \frac{2-\sec^2 x}{\sec^2 x}\, dx$$

$$= 2\int \cos^2 x - 1\, dx = 2\int \frac{1+\cos 2x}{2} - 1\, dx$$

$$= \int \cos 2x - 1\, dx = \frac{1}{2}\operatorname{sen} 2x - x$$

(h) $2\operatorname{sen}^2 \frac{x}{2} = 1 - \cos x$

$$\int \frac{dx}{\cos x - 1} = \int \frac{-1}{2\cdot \operatorname{sen}^2 \frac{x}{2}}\, dx = \int -\operatorname{cossec}^2 \frac{x}{2}\, \frac{dx}{2}$$

$$= \cot \frac{x}{2}$$

(i) $$\int x \operatorname{sen}^2 x^2\, dx = \int x\left(\frac{1-\cos 2x^2}{2}\right) dx$$

$$= \frac{x^2}{4} - \frac{1}{8}\int 4x \cos 2x^2\, dx$$

$$= \frac{x^2}{4} - \frac{\operatorname{sen} 2x^2}{8}$$

(j) $\cos[(m+n)x] = \cos mx \cdot \cos nx - \operatorname{sen} mx \cdot \operatorname{sen} nx$

$\cos[(m-n)x] = \cos mx \cdot \cos nx + \operatorname{sen} mx \cdot \operatorname{sen} nx$

$\Rightarrow \operatorname{sen} mx \cdot \operatorname{sen} nx = \dfrac{\cos[(m-n)x] - \cos[(m+n)x]}{2}$

$$\int \operatorname{sen} 3x \operatorname{sen} 6x \, dx = \frac{1}{2}\int \cos 3x - \cos 9x \, dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\operatorname{sen} 3x - \frac{1}{9}\operatorname{sen} 9x\right)$$

(k) $$\int \sec^4 \frac{x}{2} \, dx = \int \left(1 + \tan^2 \frac{x}{2}\right) \sec^2 \frac{x}{2} \, dx$$

$$= 2\left(\tan \frac{x}{2} + \frac{1}{3}\tan^3 \frac{x}{2}\right)$$

(l) $$\int \frac{\cos x + \operatorname{sen} x}{\operatorname{sen} 2x} \, dx = \int \frac{\cos x + \operatorname{sen} x}{2 \operatorname{sen} x \cos x} \, dx$$

$$= \frac{1}{2}\int \operatorname{cossec} x + \sec x \, dx$$

$$\begin{cases} \dfrac{d}{dx} \operatorname{cossec} x = -\cot x \operatorname{cossec} x \\ \dfrac{d}{dx} \cot x = -\operatorname{cossec}^2 x \end{cases}$$

$$= -\frac{1}{2}\int \frac{\operatorname{cossec} x(-\cot x - \operatorname{cossec} x)}{\operatorname{cossec} x + \cot x}\,dx$$

$$+\frac{1}{2}\int \frac{\sec x\ (\tan x + \sec x)}{\sec x + \tan x}\,dx$$

$$= \frac{1}{2}\log\left|\frac{\sec x + \tan x}{\operatorname{cossec} x + \cot x}\right|$$

Exercício 3. Mostre que se $m, n \in \mathbb{N}$ então

$$\int_{-\pi}^{\pi} \operatorname{sen} mx \operatorname{sen} nx\, dx = \begin{cases} 0 & \text{se } m \neq n \\ \pi & \text{se } m = n \end{cases}$$

$$\cos[(m+n)x] = \cos mx \cdot \cos nx - \operatorname{sen} mx \cdot \operatorname{sen} nx$$

$$\cos[(m-n)x] = \cos mx \cdot \cos nx + \operatorname{sen} mx \cdot \operatorname{sen} nx$$

$$\Rightarrow \operatorname{sen} mx \cdot \operatorname{sen} nx = \frac{\cos[(m-n)x] - \cos[(m+n)x]}{2}$$

Daí,

$$\int_{-\pi}^{\pi} \operatorname{sen} mx \operatorname{sen} nx\, dx = \int_{-\pi}^{\pi} \frac{\cos[(m-n)x] - \cos[(m+n)x]}{2}\, dx$$

Se $m \neq n$,

$$\int_{-\pi}^{\pi} \operatorname{sen} mx \operatorname{sen} nx\, dx = \frac{1}{2}\left\{\frac{\operatorname{sen}[(m-n)x]}{m-n} - \frac{\operatorname{sen}[(m+n)x]}{m+n}\right\}\Bigg|_{-\pi}^{\pi}$$

$$= 0$$

Se $m = n$,

$$\int_{-\pi}^{\pi} \operatorname{sen} mx \operatorname{sen} nx\, dx = \int_{-\pi}^{\pi} \frac{1 - \cos[(m+n)x]}{2}\, dx$$

$$= \frac{1}{2}\left\{x - \frac{\operatorname{sen}[(m+n)x]}{m+n}\right\}\Bigg|_{-\pi}^{\pi} = \pi$$

Exercício 4. Uma série finita de Fourier é dada pela soma

$$\begin{aligned} F(x) &= \sum_{n=1}^{N} a_n \operatorname{sen} nx \\ &= a_1 \operatorname{sen} x + a_2 \operatorname{sen} 2x + \ldots + a_N \operatorname{sen} Nx \end{aligned}$$

Mostre que o n-ésimo coeficiente a_n é dado pela fórmula

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \operatorname{sen} nx \, dx$$

Temos que, usando o exercício anterior,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \operatorname{sen} nx \, dx = \sum_{m=1}^{N} \frac{a_m}{\pi} \int_{-\pi}^{\pi} \operatorname{sen} mx \operatorname{sen} nx \, dx$$

$$= \frac{a_n}{\pi} \cdot \pi = a_n.$$

Exercício 5. Mostre que, para todo $\alpha \in \mathbb{R}$

$$\int_0^{\pi/2} \frac{\operatorname{sen}^\alpha x}{\operatorname{sen}^\alpha x + \cos^\alpha x}\, dx = \frac{\pi}{4}$$

(*Sugestão: mostre primeiro que* $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ *para toda função contínua* $f(x)$)

Vejamos primeiro que, $\forall f$ contínua,

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$u = a - x \Rightarrow x = a - u \Rightarrow dx = -du$

$$\int_0^a f(x)\, dx = \int_a^0 f(a-u)(-1)\, du = \int_0^a f(a-u)\, du$$

Assim,

$$I = \int_0^{\pi/2} \frac{\operatorname{sen}^\alpha x\, dx}{\operatorname{sen}^\alpha x + \cos^\alpha x} = \int_0^{\pi/2} \frac{\operatorname{sen}^\alpha(\pi/2 - x)}{\operatorname{sen}^\alpha(\pi/2 - x) + \cos^\alpha(\pi/2 - x)}\, dx$$

Lembre que

$\operatorname{sen}\left(\frac{\pi}{2} - x\right) = \cos x$, $\cos\left(\frac{\pi}{2} - x\right) = \operatorname{sen} x$

$$= \int_0^{\pi/2} \frac{\cos^\alpha x}{\operatorname{sen}^\alpha x + \cos^\alpha x}\, dx$$

Logo,

$$2I = \int_0^{\pi/2} \frac{\operatorname{sen}^\alpha x + \cos^\alpha x}{\operatorname{sen}^\alpha x + \cos^\alpha x}\, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Exercício 1. Calcule a integral usando a substituição trigonométrica indicada:

(a) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx, \quad x = 3\sec\theta$ (b) $\displaystyle\int x^3\sqrt{9-x^2}\,dx, \quad x = 3\,\text{sen}\,\theta$

(c) $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx, \quad x = 3\tan\theta$

(a) $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{\cancel{3}\cancel{\sec\theta}\,\cancel{\tan\theta}}{9\sec^{\cancel{2}}\theta\cdot\cancel{3\tan\theta}}\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\,\text{sen}\,\theta$$

(b) $x = 3\,\text{sen}\,\theta \Rightarrow dx = 3\cos\theta\,d\theta$

$$\int x^3\sqrt{9-x^2}\,dx = \int 3^3\cdot\text{sen}^3\theta\cdot 3^2\cos^2\theta\,d\theta$$

$$= 3^5\int (1-\cos^2\theta)\cdot\cos^2\theta\cdot\text{sen}\,\theta\,d\theta$$

$u = \cos\theta \Rightarrow du = -\text{sen}\,\theta\,d\theta$

$$= 3^5\int (u^2-1)u^2\,du = 3^5\left(\frac{u^5}{5} - \frac{u^3}{3}\right)$$

$x = 3\,\text{sen}\,\theta \Rightarrow u = \cos\theta = \frac{\sqrt{9-x^2}}{3}$

Logo,

$$\int x^3\sqrt{9-x^2}\,dx = 3^5\left[\frac{(9-x^2)^{5/2}}{5\cdot 3^5} - \frac{(9-x^2)^{3/2}}{3^4}\right]$$

$$= \frac{(9-x^2)^{5/2}}{5} - 3(9-x^2)^{3/2}$$

(c) $x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{3^{\cancel{4}\,3} \cdot \tan^3\theta \sec^{\cancel{2}}\theta\, d\theta}{\cancel{3} \cdot \cancel{\sec\theta}\ 1}$$

$$= 27\int (\sec^2\theta - 1) \cdot \tan\theta \sec\theta\, d\theta$$

$u = \sec\theta \Rightarrow du = \tan\theta\sec\theta\, d\theta$

$$= 27\int (u^2-1)\,du = 27\left(\frac{u^3}{3} - u\right)$$

$\frac{x}{3} = \tan\theta \Rightarrow u = \sec\theta = \frac{\sqrt{9+x^2}}{3}$

$$= 27 \cdot \left[\frac{(x^2+9)^{3/2}}{81} - \frac{(x^2+9)^{1/2}}{3}\right]$$

$$= \frac{(x^2+9)^{3/2}}{3} - 9(x^2+9)^{1/2}$$

Exercício 2. Calcule:

(a) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

(b) $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx$

(c) $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$

(d) $\displaystyle\int_2^3 \frac{dx}{(x^2-1)^{3/2}}\,dx$

(e) $\displaystyle\int_0^{1/2} x\sqrt{1-4x^2}\,dx$

(f) $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

(g) $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}\,dx$

(h) $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$

(i) $\displaystyle\int \frac{dx}{[(ax)^2-b^2]^{3/2}}$

(j) $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$

(k) $\displaystyle\int_0^2 \frac{dt}{\sqrt{4+t^2}}$

(l) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

(m) $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$

(n) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$

(o) $\displaystyle\int_0^1 \sqrt{x-x^2}\,dx$

(p) $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

(q) $\displaystyle\int x\sqrt{1-x^4}\,dx$

(r) $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-2}}$

(s) $\displaystyle\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

(a) $x = 3\,\mathrm{sen}\,\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{\cancel{3}^{2}\,3\,\mathrm{sen}^2\theta\,\cancel{\cos\theta}}{\cancel{3}\,\cancel{\cos\theta}}\,d\theta$$

$\cos 2\theta = \cos^2\theta - \mathrm{sen}^2\theta = 1 - 2\,\mathrm{sen}^2\theta \Rightarrow \mathrm{sen}^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \frac{3}{2}\int 1 - \cos 2\theta\, d\theta = \frac{3}{2}\left(\theta - \frac{\mathrm{sen}\,2\theta}{2}\right)$$

$$= \frac{3}{2}\theta - \frac{3}{2}\,\mathrm{sen}\,\theta\sqrt{1-\mathrm{sen}^2\theta}$$

$$= \frac{3}{2}\arcsin\frac{x}{3} - \frac{\cancel{3}}{2}\,\frac{x}{\cancel{3}}\cdot\frac{\sqrt{9-x^2}}{3}$$

$$= \frac{3}{2}\arcsin\frac{x}{3} - \frac{x}{6}\sqrt{9-x^2}$$

(b) $x = \sec\theta \Rightarrow dx = \sec\theta \tan\theta \, d\theta$

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan^2\theta \, \cancel{\sec\theta}}{\sec^{\cancel{4}3}\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \cos^{\cancel{3}}\theta \, d\theta = \frac{1}{3}\sin^3\theta$$

$$= \frac{1}{3}\left(1-\cos^2\theta\right)^{3/2} = \frac{1}{3}\left(1-\frac{1}{x^2}\right)^{3/2}$$

$$= \frac{(x^2-1)^{3/2}}{3x^3}$$

(c) $x = a\tan\theta \Rightarrow dx = a\sec^2\theta \, d\theta$

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta \, d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\pi/4} \cos\theta \, d\theta$$

$$= \frac{1}{a^2}\sin\theta \Big|_0^{\pi/4} = \frac{1}{\sqrt{2}\,a^2}$$

(d) $x = \sec\theta \Rightarrow dx = \sec\theta \tan\theta \, d\theta$

$$\int_2^3 \frac{dx}{(x^2-1)^{3/2}} = \int_{\text{arcsec}\,2}^{\text{arcsec}\,3} \frac{\sec\theta \, \cancel{\tan\theta}}{\tan^{\cancel{3}2}\theta}\,d\theta$$

$$= \int_{\text{arcsec}\,2}^{\text{arcsec}\,3} \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\cdot\frac{1}{\cancel{\cos\theta}}\,d\theta = -\frac{1}{\sin\theta}\Bigg|_{\text{arc}\sec 2}^{\text{arc}\sec 3}$$

$$\theta = \text{arc sec } x \Rightarrow \sec\theta = \frac{1}{\cos\theta} = x$$

$$\Rightarrow \cos\theta = \frac{1}{x} \Rightarrow \text{sen}\,\theta = \frac{\sqrt{x^2-1}}{x}$$

Logo,

$$\int_2^3 \frac{dx}{(x^2-1)^{3/2}} = \frac{x}{\sqrt{x^2-1}}\Bigg|_3^2 = 2 - \frac{3}{2\sqrt{2}}$$

(e) $$\int_0^{1/2} x\sqrt{1-4x^2}\,dx = -\frac{1}{8}\cdot\frac{2}{3}\cdot(1-4x^2)^{3/2}\Bigg|_0^{1/2}$$

$$= \frac{1}{12}$$

(f) $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{x^2-9}}{x^3}dx = \int \frac{\cancel{3}^2\tan^2\theta\,\cancel{\sec\theta}}{3^{\cancel{3}}\cdot\sec^{\cancel{3}2}\theta}\,d\theta$$

$$= \frac{1}{3}\int \frac{\text{sen}^2\theta}{\cancel{\cos^2\theta}}\,\cancel{\cos^2\theta}\,d\theta$$

$$= \frac{1}{3}\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{3}\left(\frac{\theta}{2} - \frac{1}{4}\text{sen}\,2\theta\right)$$

$$= \frac{\theta}{6} - \frac{1}{6}\sqrt{1-\cos^2\theta}\cdot\cos\theta$$

$x = \frac{3}{\cos\theta} \Rightarrow \cos\theta = \frac{3}{x} \Rightarrow \theta = \arccos\frac{3}{x}$

$$= \frac{1}{6}\arccos\frac{3}{x} - \frac{1}{6}\,\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x}$$

$$= \frac{1}{6}\arccos\frac{3}{x} - \frac{1}{2}\,\frac{\sqrt{x^2-9}}{x^2}$$

(g) $x = a\,\text{sen}\,\theta \Rightarrow dx = a\cos\theta\, d\theta$

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\text{sen}^2\theta\cdot a^2\cos^2\theta\, d\theta$$

$$= a^4\int_0^{\pi/2}\frac{\text{sen}^2 2\theta}{4}\,d\theta$$

$\cos^2 u - \text{sen}^2 u = 1 - 2\,\text{sen}^2 u = \cos 2u$

$\Rightarrow \text{sen}^2 u = \dfrac{1-\cos 2u}{2}$

$$= \frac{a^4}{8}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta = \frac{a^4}{8}\left(\theta - \frac{\text{sen}\,4\theta}{4}\right)\Big|_0^{\pi/2}$$

$$= \frac{a^4}{8}\,\frac{\pi}{2} = \frac{\pi a^4}{16}$$

(h) $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \sqrt{x^2-7}$

(i) $ax = b\sec\theta \Rightarrow dx = \frac{b}{a}\tan\theta\sec\theta\, d\theta$

$$\int \frac{dx}{[(ax)^2-b^2]^{3/2}} = \int \frac{\cancel{b}\,\cancel{\tan\theta}\sec\theta \; d\theta}{ab^{3\,2}\tan^{3\,2}\theta}$$

$$= \frac{1}{ab^2}\int \frac{\cos^{\cancel{2}}\theta \; d\theta}{\operatorname{sen}^2\theta\,\cancel{\cos\theta}} = \frac{-1}{ab^2}\cdot\frac{1}{\operatorname{sen}\theta}$$

$$= \frac{-1}{ab^2}\cdot\frac{1}{\sqrt{1-\cos^2\theta}}$$

$$ax = b\sec\theta = \frac{b}{\cos\theta} \Rightarrow \cos\theta = \frac{b}{ax}$$

$$= \frac{-1}{ab^2}\cdot\frac{ax}{\sqrt{(ax)^2-b^2}} = \frac{-x}{b^2\sqrt{(ax)^2-b^2}}$$

(j) $3x = \sec\theta \Rightarrow dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$

$\frac{1}{\cos\theta} = 3x \Rightarrow \cos\theta = 1/3x$

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\pi/4}^{\pi/3} \frac{3^{-1}\cancel{\sec\theta}\,\cancel{\tan\theta}\; d\theta}{3^{-5}\sec^{\cancel{5}\,4}\theta\,\cancel{\tan\theta}} = \int_{\pi/4}^{\pi/3} 3^4\cdot\cos^4\theta\, d\theta$$

$$2\cos^2\theta - 1 = \cos^2\theta - \operatorname{sen}^2\theta = \cos 2\theta$$

$$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 3^4\int_{\pi/4}^{\pi/3}\left(\frac{1+\cos 2\theta}{2^2}\right)^2 d\theta = \frac{3^4}{2^2}\int_{\pi/4}^{\pi/3} 1 + 2\cos 2\theta + \cos^2 2\theta \quad d\theta$$

$$= \frac{3^4}{2^2}\int_{\pi/4}^{\pi/3} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{3^4}{2^2}\left[\theta + \operatorname{sen} 2\theta + \frac{\theta}{2} + \frac{1}{8}\operatorname{sen} 4\theta\right]\Bigg|_{\pi/4}^{\pi/3}$$

$$= \frac{3^4}{2^2}\left[\frac{3}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{\sqrt{3}}{2} - 1 + \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{3^4}{2^2}\left[\frac{\cancel{3}}{2}\cdot\frac{\pi}{\cancel{3}\cdot 4} + \frac{7}{8}\cdot\frac{\sqrt{3}}{2} - 1\right]$$

$$= \frac{3^4}{2^2}\cdot\left(\frac{\pi}{8} + \frac{7\sqrt{3}}{16} - 1\right)$$

(K) $t = 2\tan\theta \Rightarrow dt = 2\sec^2\theta\, d\theta$

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^{\pi/4} \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{2}\,\cancel{\sec\theta}}\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta(\sec\theta + \tan\theta)}{\tan\theta + \sec\theta}\, d\theta = \log(\sec\theta + \tan\theta)\Big|_0^{\pi/4}$$

$$= \log(\sqrt{2} + 1)$$

(l) $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\cancel{\sec^2\theta}\, d\theta}{\sec^{\cancel{4}\,2}\theta \, d\theta} = \int_0^{\pi/4} \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{1+\cos 2\theta}{2} \, d\theta = \frac{\theta}{2} + \frac{\operatorname{sen} 2\theta}{4} \Big|_0^{\pi/4}$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

(m) $x = \operatorname{senh} u \Rightarrow dx = \cosh u \, du$

$$\int \frac{\sqrt{1+x^2}}{x} \, dx = \int \frac{\cosh^2 u}{\operatorname{senh} u} \, du$$

$$= \int \frac{1+\operatorname{senh}^2 u}{\operatorname{senh} u} \, du = \int \operatorname{cossech} u + \operatorname{senh} u \, du$$

$$= \int \operatorname{cossech} u \, du + \cosh u$$

$$\begin{cases} \dfrac{d}{du} \operatorname{cossech} u = \dfrac{d}{du} \dfrac{1}{\operatorname{senh} u} = -\coth u \cdot \operatorname{cossech} u \\[2ex] \dfrac{d}{du} \coth u = \dfrac{d}{du} \dfrac{\cosh u}{\operatorname{senh} u} = -\operatorname{cossech}^2 u \end{cases}$$

Assim,

$$\int \operatorname{cossech} u\, du = -\int \frac{\operatorname{cossech}(-\coth u - \operatorname{cossech} u)\, du}{\operatorname{cossech} u + \coth u}$$

$$= -\log\left|\operatorname{cossech} u + \coth u\right|$$

Logo, para $x = \operatorname{senh} u$

$$\int \frac{\sqrt{1+x^2}}{x}\, dx = \cosh u - \log\left|\operatorname{cossech} u + \coth u\right|$$

$$= \sqrt{1+\operatorname{senh}^2 u} - \log\left|\frac{1}{\operatorname{senh} u} + \frac{\sqrt{1+\operatorname{senh}^2 u}}{\operatorname{senh} u}\right|$$

$$= \sqrt{1+x^2} + \log\left|\frac{x}{1+\sqrt{1+x^2}}\right|$$

(h) $x^2 + 2x + 5 = (x+1)^2 + 2^2$

$x + 1 = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{\cancel{2}\,\cancel{\sec\theta}}\, d\theta = \int \sec\theta\, d\theta$$

$$= \int \frac{\sec\theta(\tan\theta + \sec\theta)}{\sec\theta + \tan\theta}\, d\theta = \log\left|\sec\theta + \tan\theta\right|$$

$$= \log\left|\tan\theta + \sqrt{\tan^2\theta + 1}\right|$$

$$= \log \left| \frac{x+1}{2} + \frac{\sqrt{(x+1)^2+4}}{2} \right|$$

$$= \log \left(\frac{x+1+\sqrt{x^2+2x+5}}{2} \right)$$

(o) $\sqrt{x} = \operatorname{sen}\theta \Rightarrow x = \operatorname{sen}^2\theta$

$\Rightarrow dx = 2\operatorname{sen}\theta \cos\theta \, d\theta$

$$\int \frac{dx}{\sqrt{x-x^2}} = \int \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int \frac{2\cancel{\operatorname{sen}\theta}\,\cancel{\cos\theta}}{\cancel{\operatorname{sen}\theta}\,\cancel{\cos\theta}} \, d\theta$$

$$= 2\theta = 2 \operatorname{arc\,sen} \sqrt{x}$$

(p) $\displaystyle\int \frac{x^2+1}{(x^2-2x+2)^2} dx = \int \frac{x^2+1}{[(x-1)^2+1]^2} dx$

$x - 1 = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$= \int \frac{(\tan\theta+1)^2+1}{\sec^{\cancel{4}\,2}\theta} \cdot \cancel{\sec^2\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta + 2\tan\theta + 1}{\sec^2\theta} \, d\theta$$

$$= \int 1 + 2\frac{\operatorname{sen}\theta}{\cancel{\cos\theta}} \cos^{\cancel{2}}\theta + \cos^2\theta \, d\theta$$

$$= \theta + \operatorname{sen}^2\theta + \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \theta + \operatorname{sen}^2\theta + \frac{\theta}{2} + \frac{\operatorname{sen} 2\theta}{4}$$

$$= \frac{3}{2}\theta + \frac{1-\cos 2\theta}{2} + \frac{\operatorname{sen} 2\theta}{4}$$

(podemos desprezar a constante 1/2 pois queremos apenas uma primitiva)

Lembre que

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}, \quad \operatorname{sen} 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \frac{3}{2}\operatorname{arc\,tan}(x-1) - \frac{1}{2}\cdot\frac{1-(x-1)^2}{1+(x-1)^2}$$

$$+ \frac{1}{\cancel{4}\,2}\cdot\frac{\cancel{2}(x-1)}{1+(x-1)^2}$$

$$= \frac{3}{2}\operatorname{arc\,tan}(x-1) + \frac{1}{2}\,\frac{x^2-2x+1-1+x-1}{x^2-2x+2}$$

$$= \frac{3}{2}\operatorname{arc\,tan}(x-1) + \frac{1}{2}\,\frac{x^2-x-1}{x^2-2x+2}$$

(9) $x^2 = \operatorname{sen}\theta \Rightarrow x\,dx = \frac{1}{2}\cos\theta\, d\theta$

$$\int x\sqrt{1-x^4}\,dx = \int \frac{\cos^2\theta}{2}\,d\theta$$

$$= \int \frac{1+\cos 2\theta}{4}\,d\theta = \frac{\theta}{4} + \frac{\operatorname{sen} 2\theta}{8}$$

$$= \frac{\theta}{4} + \frac{\operatorname{sen}\theta\sqrt{1-\operatorname{sen}^2\theta}}{4}$$

$$= \frac{1}{4}\operatorname{arc\,sen} x^2 + \frac{x^2\sqrt{1-x^4}}{4}$$

(r) $x = \sqrt{2}\sec\theta \Rightarrow dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$

$$\int \frac{dx}{x^4\sqrt{x^2-2}} = \int \frac{\cancel{\sqrt{2}}\sec\theta\,\cancel{\tan\theta}}{2^2\cdot\sec^4\theta\,\cancel{\sqrt{2}}\,\cancel{\tan\theta}}\,d\theta$$

$$= \frac{1}{4}\int \cos^3\theta\,d\theta = \frac{1}{4}\int(1-\operatorname{sen}^2\theta)\cos\theta\,d\theta$$

$$= \frac{1}{4}\int \cos\theta - \operatorname{sen}^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{4}\left(\operatorname{sen}\theta - \frac{1}{3}\operatorname{sen}^3\theta\right)$$

$$\frac{1}{\cos\theta} = \frac{x}{\sqrt{2}} \Rightarrow \cos\theta = \frac{\sqrt{2}}{x} \Rightarrow \operatorname{sen}\theta = \sqrt{\frac{x^2-2}{x}}$$

$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{2} - \frac{1}{3}\frac{(x^2-2)^{3/2}}{x^3}\right]$$

(5) $\sqrt{1-x^2} = 2\,\text{sen}^2\theta$

$$\Rightarrow \frac{-x\,dx}{\sqrt{1-x^2}} = 4\,\text{sen}\,\theta\cos\theta\,d\theta$$

$$\Rightarrow dx = -\frac{\sqrt{1-x^2}}{x}\cdot 4\,\text{sen}\,\theta\cos\theta\,d\theta$$

$$= -\frac{8\,\text{sen}^3\theta\cos\theta\,d\theta}{x}$$

$$\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx = \int_{\pi/4}^{0} \cancel{x}\sqrt{2}\cos\theta\left(-\frac{8\,\text{sen}^3\theta\cos\theta}{\cancel{x}}\right)d\theta$$

$$= 8\sqrt{2}\int_0^{\pi/4} \text{sen}^3\theta\cos^2\theta\,d\theta$$

$$= 8\sqrt{2}\int_0^{\pi/4}(1-\cos^2\theta)\cos^2\theta\cdot\text{sen}\,\theta\,d\theta$$

$$= 8\sqrt{2}\int_0^{\pi/4}\cos^2\theta\,\text{sen}\,\theta - \cos^4\theta\cdot\text{sen}\,\theta\,d\theta$$

$$= 8\sqrt{2}\left(\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi/4}$$

$$= 8\sqrt{2}\left(\frac{1}{20\sqrt{2}} - \frac{1}{6\sqrt{2}} - \frac{1}{5} + \frac{1}{3}\right)$$

$$= \frac{2}{5} - \frac{4}{3} - \frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3}$$

$$= \frac{6-20}{15} - \left(\frac{3-5}{15}\right)8\sqrt{2} = \frac{16\sqrt{2}-14}{15}$$

Exercício 3. Use a substituição hiperbólica $x = a \operatorname{senh} t$ para mostrar que

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{ar\,senh}\left(\frac{x}{a}\right)$$

Em seguida, use a substituição trigonométrica $x = a \tan t$ na mesma integral para obter

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2 + a^2})$$

$x = a \operatorname{senh} t \Rightarrow dx = a \cosh t \, dt$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t}{a \cosh t} dt = t = \operatorname{ar\,senh} \frac{x}{a}$$

$x = a \tan t \Rightarrow dx = a \sec^2 t \, dt$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{\cancel{a}\sec^{\cancel{2}} t}{\cancel{a}\,\cancel{\sec t}} dt = \int \sec t \, dt$$

$$= \int \frac{\sec t \, (\sec t + \tan t)}{\tan t + \sec t} dt$$

$$= \log|\tan t + \sec t|$$

$$= \log|\tan t + \sqrt{1 + \tan^2 t}|$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{a^2 + x^2}{a}}\right|$$

$$= \log(x + \sqrt{x^2 + a^2}) - \underbrace{\log a}$$

Como queremos uma primitiva, a constante pode ser desprezada

Exercício 4. Calcule, primeiro usando substituição trigonométrica e depois hiperbólica, a expressão

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx$$

$$x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$$

$$\int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx = \int \frac{\cancel{a^2}\tan^2\theta}{\cancel{a^3}\sec^3\theta}\,\cancel{a}\,\cancel{\sec^2\theta}\,d\theta$$

$$= \int \frac{\operatorname{sen}^2\theta}{\cos^{\cancel{2}}\theta}\cdot\cancel{\cos\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \int \sec\theta - \cos\theta\,d\theta$$

$$= \log|\sec\theta + \tan\theta| - \operatorname{sen}\theta$$

$$\operatorname{sen}\theta = \frac{\operatorname{sen}\theta}{\cos\theta}\cdot\cos\theta = \frac{\tan\theta}{\sec\theta}$$

$$= \log\left|\tan\theta + \sqrt{1+\tan^2\theta}\right| - \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \log\left(\frac{x}{a} + \frac{\sqrt{a^2+x^2}}{a}\right) - \frac{1}{a}\,\frac{x}{\frac{\sqrt{a^2+x^2}}{a}}$$

$$= \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) - \frac{x}{\sqrt{a^2+x^2}}$$

Agora com substituição hiperbólica:

$$x = a\,\mathrm{senh}\,u \Rightarrow dx = a\cosh u\,du$$

$$\int \frac{x^2dx}{(x^2+a^2)^{3/2}} = \int \frac{\cancel{a^2}\,\mathrm{senh}^2u\,\cancel{a}\,\cancel{\cosh u}}{\cancel{a^3}\cosh^{\cancel{3}\,2}u}\,du$$

$$= \int \tanh^2 u\,du$$

$$\tanh^2u - 1 = \frac{\mathrm{senh}^2u - \cosh^2u}{\cosh^2u} = -\mathrm{sech}^2u$$

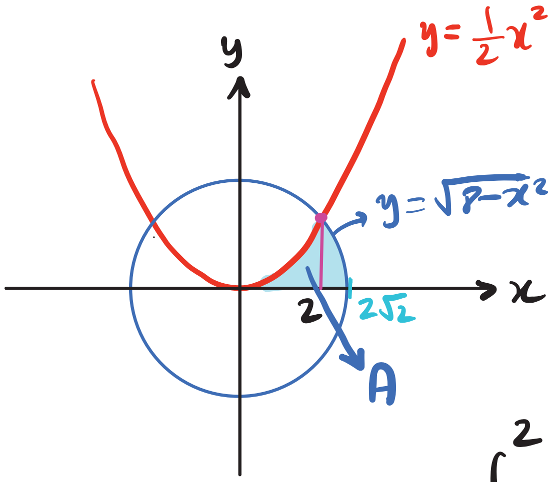
$$\Rightarrow \tanh^2u = 1 - \mathrm{sech}^2u$$

$$\frac{d}{du}\tanh u = \frac{\cosh^2u - \mathrm{senh}^2u}{\cosh^2u} = \mathrm{sech}^2u$$

$$= \int 1 - \mathrm{sech}^2u\,du = u - \tanh u$$

$$= u - \frac{\mathrm{senh}\,u}{\sqrt{1+\mathrm{senh}^2u}} = \mathrm{arsenh}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{x^2+a^2}}$$

Exercício 5. A parábola $y = x^2/2$ divide o disco $x^2 + y^2 \leqslant 8$ em duas partes. Calcule a área de ambas as partes.



$$\left(\frac{1}{2}x^2\right)^2 + x^2 = \frac{x^4}{4} + x^2 = 8$$

$$x^4 + 4x^2 - 32 = 0$$

$$x^2 = \frac{-4 + \sqrt{16+128}}{2} = \frac{-4+12}{2} = 4$$

$$\Rightarrow x = 2$$

$$\text{ÁREA AZUL} = A = \int_0^2 \frac{1}{2}x^2\,dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2}\,dx$$

$$\begin{cases} x = 2\sqrt{2}\,\text{sen}\,\theta \\ dx = 2\sqrt{2}\cos\theta\,d\theta \end{cases}$$

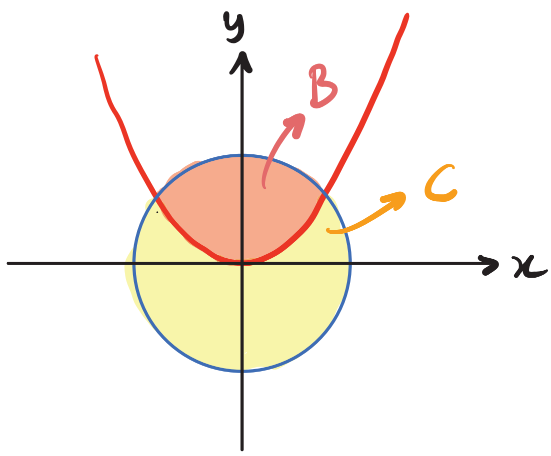
$$\Rightarrow A = \frac{1}{6}2^3 + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 8\cos^2\theta\,d\theta$$

$$\begin{cases} \cos 2\theta = 2\cos^2\theta - 1 \\ \cos^2\theta = \dfrac{1+\cos 2\theta}{2} \end{cases}$$

$$= \frac{4}{3} + 8\int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{4}{3} + 4\left(\theta + \frac{1}{2}\,\text{sen}\,2\theta\right)\Bigg|_{\pi/4}^{2\pi/4}$$

$$= \frac{4}{3} + 4\cdot\left(\frac{\pi}{4} - \frac{1}{2}\right) = \pi - \frac{2}{3}$$

Logo:



$$C = \frac{8\pi}{2} + 2\left(\pi - \frac{2}{3}\right) = 6\pi - \frac{4}{3}$$

$$B = \frac{8\pi}{2} - 2\left(\pi - \frac{2}{3}\right) = 2\pi + \frac{4}{3}$$

Exercício 1. Escreva na forma de decomposição em frações parciais. Não é necessário determinar os coeficientes.

(a) $\dfrac{4+x}{(1+2x)(3-x)}$ (b) $\dfrac{x-6}{x^2+x-6}$ (c) $\dfrac{1}{x^2+x^4}$

(d) $\dfrac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}$ (e) $\dfrac{x^4+1}{x^5+4x^3}$ (f) $\dfrac{t^6+1}{t^6+t^3}$

(g) $\dfrac{1-x}{x^3+x^4}$ (h) $\dfrac{x^2}{x^2+x+6}$ (i) $\dfrac{x^3+1}{x^3-3x^2-2x}$

(j) $\dfrac{x^2-2}{x^3+2x^2-1}$ (k) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$

(a) $\dfrac{a}{1+2x} + \dfrac{b}{3-x}$

(b) $\dfrac{x-6}{x^2+x-6} = \dfrac{x-6}{(x-2)(x+3)} = \dfrac{a}{x-2} + \dfrac{b}{x+3}$

(c) $\dfrac{1}{x^2+x^4} = \dfrac{1}{x^2(1+x^2)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{dx+e}{1+x^2}$

(d) Primeiro passo é dividir os polinômios:

$$
\begin{array}{l|l}
\cancel{x^4} - 2x^3 + \cancel{x^2} + 2x - 1 & x^2 - 2x + 1 \\
\hline
-\cancel{x^4} + 3x^3 - \cancel{x^2} & x^2 + x + 2 \\
\quad \cancel{x^3} + 2x - 1 & \\
\quad -\cancel{x^3} + 2x^2 - x & \\
\qquad \cancel{2x^2} + x - 1 & \\
\qquad -\cancel{2x^2} + 4x - 2 & \\
\qquad\quad 5x - 3 &
\end{array}
$$

Logo:

$x^4 - 2x^3 + x^2 + 2x - 1$
$= (x^2+x+2)\cdot(x^2-2x+1)$
$+ 5x - 3$

Então

$$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2+x+2 + \frac{5x-3}{x^2-2x+1}$$

$$= x^2+x+2 + \frac{5x-3}{(x-1)^2}$$

$$= x^2+x+2 + \frac{a}{x-1} + \frac{b}{(x-1)^2}$$

(e) $$\frac{x^4+1}{x^5+4x^3} = \frac{x^4+1}{x^3(x^2+4)}$$

$$= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{dx+e}{x^2+4}$$

(f) $$\frac{t^6+1}{t^6+t^3} = \frac{t^6+t^3-t^3+1}{t^6+t^3}$$

$$= 1 + \frac{1-t^3}{t^3(t^3+1)} = 1 + \frac{1-t^3}{t^3(t+1)(t^2-t+1)}$$

$$= 1 + \frac{a}{t} + \frac{b}{t^2} + \frac{c}{t^3} + \frac{d}{t+1} + \frac{et+f}{t^2-t+1}$$

(g) $$\frac{1-x}{x^3+x^4} = \frac{1-x}{x^3(1+x)}$$

$$= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{1+x}$$

(h) $$\frac{x^2}{x^2+x+6} = \frac{x^2+x+6-(x+6)}{x^2+x+6}$$

$$= 1 - \frac{x+6}{x^2+x+6} = 1 + \frac{ax+b}{x^2+x+6}$$

(i) $$\frac{x^3+1}{x^3-3x^2-2x} = \frac{x^3-3x^2-2x+3x^2+2x+1}{x^3-3x^2-2x}$$

$$= 1 + \frac{3x^2+2x+1}{x^3-3x^2-2x} = 1 + \frac{3x^2+2x+1}{x(x^2-3x-2)}$$

As raízes de x^2-3x-2 são

$$x = \frac{3 \pm \sqrt{17}}{2}$$

Logo,

$$\frac{x^3+1}{x^3-3x^2-2x} = 1 + \frac{a}{x} + \frac{b}{x-\frac{3-\sqrt{17}}{2}} + \frac{c}{x-\frac{3+\sqrt{17}}{2}}$$

(j) -1 é raiz de x^3+2x^2-1

$\Rightarrow$ $x+1$ divide x^3+2x^2-1

$$\begin{array}{r|l} \cancel{x^3}+2x^2-1 & x+1 \\ \hline -\cancel{x^3}-x^2 & x^2+x-1 \\ \hline x^2-1 & \\ \vdots & \\ 0 & \end{array}$$

As raízes de x^2+x-1 são

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Logo,

$$\frac{x^2-2}{x^3+2x^2-1} = \frac{a}{x+1} + \frac{b}{x+\frac{1-\sqrt{5}}{2}} + \frac{c}{x+\frac{1+\sqrt{5}}{2}}$$

(k) $$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \frac{x^5+1}{x(x-1)(x^2+1)^2}$$

$$= \frac{a}{x} + \frac{b}{x-1} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2}$$

Exercício 2. Calcule a integral:

(a) $\displaystyle\int \frac{x}{x-6}\,dx$ (b) $\displaystyle\int \frac{x^2}{x+4}\,dx$ (c) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

(d) $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$ (e) $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$ (f) $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dx$

(g) $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$ (h) $\displaystyle\int \frac{1}{(t+4)(t-1)}\,dt$ (i) $\displaystyle\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx$

(j) $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx$ (k) $\displaystyle\int \frac{x^3+4x+3}{x^4+5x^2+4}\,dx$ (l) $\displaystyle\int \frac{1}{x^3-1}\,dx$

(m) $\displaystyle\int \frac{x^3+2x}{x^4+4x^2+3}\,dx$ (n) $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$ (o) $\displaystyle\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx$

(a) $$\int \frac{x}{x-6}\,dx = \int \frac{x-6+6}{x-6}\,dx = \int 1 + \frac{6}{x-6}\,dx$$

$$= x + 6\log|x-6|$$

(b) $$\int \frac{x^2}{x+4}\,dx = \int \frac{(x^2+4x)-(4x+16)+16}{x+4}\,dx$$

$$= \int x - 4 + \frac{16}{x+4}\,dx$$

$$= \frac{1}{2}x^2 - 4x + 16\log|x+4|$$

(c) $$\frac{x-9}{(x+5)(x-2)} = \frac{a}{x+5} + \frac{b}{x-2}$$

$$= \frac{(a+b)x + (5b-2a)}{(x+5)(x-2)}$$

$$\Rightarrow \begin{cases} a+b=1 \Rightarrow b = 1-a \\ 2a-5b = 9 \Rightarrow 2a-5+5a = 9 \end{cases}$$

$$\Rightarrow 7a = 14 \Rightarrow a = 2,\ b = -1$$

Logo,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \frac{2}{x+5} - \frac{1}{x-2}\,dx$$

$$= \log \frac{(x+5)^2}{|x-2|}$$

(d) As raízes de $2x^2 + 3x + 1$ são

$$x = \frac{-3 \pm 1}{4} = -1 \text{ ou } -\frac{1}{2}$$

$$\Rightarrow 2x^2 + 3x + 1 = 2\left(x + \frac{1}{2}\right)(x+1)$$

$$\Rightarrow \frac{2}{2x^2+3x+1} = \frac{1}{\left(x+\frac{1}{2}\right)(x+1)} = \frac{a}{x+\frac{1}{2}} + \frac{b}{x+1}$$

$$= \frac{(a+b)x + \left(\frac{b}{2} + a\right)}{\left(x+\frac{1}{2}\right)\cdot(x+1)}$$

$$\Rightarrow \begin{cases} a+b=0 \Rightarrow b=-a \\ \frac{b}{2}+a=1 \Rightarrow \frac{a}{2}=1 \Rightarrow a=2, \quad b=-2 \end{cases}$$

Logo,

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \frac{2}{x+\frac{1}{2}} - \frac{2}{x+1}\,dx$$

$$= 2\log\left(\frac{x+1/2}{x+1}\right)\Big|_0^1 = 2\left[\log\frac{3}{4} - \log\frac{1}{2}\right]$$

$$= 2\log\frac{3}{2}$$

(e) $$\frac{x^3-2x^2-4}{x^3-2x^2} = 1 - \frac{4}{x^2(x-2)}$$

Temos

$$\frac{1}{x^2(x-2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-2}$$

$$= \frac{ax^2-2ax+bx-2b+cx^2}{x^2(x-2)}$$

$$\Rightarrow \begin{cases} a+c=0 \Rightarrow c=-a=1/4 \\ b-2a=0 \Rightarrow a=b/2=-1/4 \\ -2b=1 \Rightarrow b=\frac{-1}{2} \end{cases}$$

Logo,

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int_3^4 1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2}\,dx$$

$$= x + \log x - \frac{2}{x} - \log(x-2)\Big|_3^4$$

$$= 1 + \cancel{2}\log 2 - \log 3 - \frac{1}{2} + \frac{2}{3} - \cancel{\log 2}$$

$$= \frac{7}{6} - \log \frac{3}{2}$$

(f) $\dfrac{4y^2 - 7y - 12}{y(y+2)(y-3)} = \dfrac{a}{y} + \dfrac{b}{y+2} + \dfrac{c}{y-3}$

$$= \frac{ay^2 - ay - 6a + by^2 - 3by + cy^2 + 2cy}{y(y+2)(y-3)}$$

$$\Rightarrow \begin{cases} a + b + c = 4 \Rightarrow b + c = 2 \Rightarrow c = 2 - b \\ a + 3b - 2c = 7 \Rightarrow 3b - 4 + 2b = 5 \Rightarrow 5b = 9 \\ \qquad \qquad \qquad \qquad \qquad \qquad \Rightarrow b = 9/5 \\ 6a = 12 \Rightarrow a = 2 \qquad \qquad \qquad \Rightarrow c = 1/5 \end{cases}$$

Logo,

$$\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\,dy = \int_1^2 \frac{2}{y} + \frac{9}{5}\cdot\frac{1}{y+2} + \frac{1}{5}\cdot\frac{1}{y-3}\,dy$$

$$= 2\log y + \frac{9}{5}\log(y+2) + \frac{1}{5}\log(3-y)\Big|_1^2$$

$$= 2\log 2 + \frac{18}{5}\log 2 - \frac{9}{5}\log 3 - \frac{1}{5}\log 2$$

$$= \frac{27}{5}\log 2 - \frac{9}{5}\log 3$$

$$= \frac{9}{5}\log\frac{8}{3}$$

(g) $$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x+2}$$

$$= \frac{ax^2+3ax+2a+bx+2b+cx^2+2cx+c}{(x+1)^2(x+2)}$$

$$\Rightarrow \begin{cases} a+c=1 \Rightarrow c=1-a \\ 3a+b+2c=1 \Rightarrow 3a+b+2-2a=1 \\ \qquad \Rightarrow b=-1-a \\ 2a+2b+c=1 \Rightarrow 2a-2-2a+1-a=1 \end{cases}$$

$$\Rightarrow a=-2$$

$$\Rightarrow b=1, \quad c=3$$

Logo,

$$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int_0^1 \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\,dx$$

$$= -2\log(x+1) - \frac{1}{x+1} + 3\log(x+2)\Big|_0^1$$

$$= -2\log 2 - \frac{1}{2} + 1 + 3\log 3 - 3\log 2$$

$$= 3\log 3 - 5\log 2 + \frac{1}{2}$$

(h) $$\frac{1}{(t+4)(t-1)} = \frac{a}{t+4} + \frac{b}{t-1} = \frac{(a+b)t + 4b-a}{(t+4)(t+1)}$$

$$\Rightarrow \begin{cases} a+b=0 \Rightarrow b=-a \\ 4b-a=1 \Rightarrow -5a=1 \Rightarrow a=-1/5 \\ \qquad \Rightarrow b=1/5 \end{cases}$$

Logo,

$$\int \frac{1}{(t+4)(t-1)}\,dt = \frac{1}{5}\int \frac{1}{t-1} - \frac{1}{t+4}\,dt$$

$$= \frac{1}{5}\left(\log|t-1| - \log|t+4|\right)$$

$$= \frac{1}{5}\log\frac{|t-1|}{|t+4|}$$

$(x-1)(3x-1)$

$3x^2-4x+1$

(i) $$\frac{-5x^2+3x}{(3x-1)(x-1)^2} = \frac{a}{3x-1} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$$

$$= \frac{ax^2-2ax+a+3bx^2-4bx+b+3cx-c}{(3x-1)(x-1)^2}$$

$$\Rightarrow \begin{cases} a+b=-5 \Rightarrow b=-5-a \\ -2a-4b+3c=3 \\ a+b-c=0 \Rightarrow c=-5 \end{cases}$$

$\Rightarrow -2a+20+4a-15=3$

$\Rightarrow 2a=-2 \Rightarrow a=-1$

$\Rightarrow b=-4$

Logo,

$$\int_2^3 \frac{-5x^2+3x}{(3x-1)(x-1)^2}\,dx = -\int_2^3 \frac{1}{3x-1} + \frac{4}{x-1} + \frac{5}{(x-1)^2}\,dx$$

$$= -\log(3x-1) - 4\log(x-1) + \frac{5}{x-1}\Big|_2^3$$

$$= -3\log 2 + \log 5 - 4\log 2 + \frac{5}{2} - 5$$

$$= \log 5 - 7\log 2 - \frac{5}{2}$$

(j) Note que -1 é raiz de $x^3 + x^2 + x + 1$.
Dividindo por $x+1$, vem

$$\begin{array}{rl|l} x^3 + x^2 + x + 1 & & x+1 \\ -x^3 - x^2 & & x^2+1 \\ \hline x+1 & & \\ 0 & & \end{array}$$

Logo,

$$\frac{4x}{x^3+x^2+x+1} = \frac{4x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$

$$= \frac{ax^2 + a + bx^2 + cx + bx + c}{(x+1)(x^2+1)}$$

$$\Rightarrow \begin{cases} a+b=0 \Rightarrow b=-a \\ b+c=4 \\ a+c=0 \Rightarrow c=-a \end{cases}$$

$$\Rightarrow -2a = 4 \Rightarrow \begin{cases} a=-2 \\ b=c=2 \end{cases}$$

Assim,

$$\int \frac{4x}{x^3+x^2+x+1}\,dx = \int \frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\,dx$$

$$= -2\log|x+1| + \log(x^2+1) + 2\arctan x$$

(k) $$\frac{x^3+4x+3}{x^4+5x^2+4} = \frac{x^3+4x+3}{(x^2+1)(x^2+4)}$$

$$= \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+4}$$

$$= \frac{ax^3+bx^2+4ax+4b+cx^3+dx^2+cx+d}{(x^2+1)(x^2+4)}$$

$$\Rightarrow \begin{cases} a+c=1 \Rightarrow c=1-a \\ b+d=0 \Rightarrow d=-b \\ 4a+c=4 \Rightarrow 4a+1-a=4 \Rightarrow a=1 \Rightarrow c=0 \\ 4b+d=3 \Rightarrow 4b-b=3 \Rightarrow b=1 \Rightarrow d=-1 \end{cases}$$

Logo,

$$\int \frac{x^3+4x+3}{x^4+5x^2+4}\,dx = \int \frac{x+1}{x^2+1} - \frac{1}{x^2+4}\,dx$$

$$= \int \frac{1}{2}\cdot\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{1}{x^2+2^2}\,dx$$

$$= \frac{1}{2}\log(x^2+1) + \arctan x - \frac{1}{2}\arctan\left(\frac{x}{2}\right)$$

(l) $$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}$$

$$= \frac{ax^2 + ax + a + bx^2 + (c-b)x - c}{x^3 - 1}$$

$$\Rightarrow \begin{cases} a + b = 0 \Rightarrow b = -a \\ a - b + c = 0 \\ a - c = 1 \Rightarrow a = 1 + c \end{cases}$$

$$\Rightarrow 1 + c + 1 + c + c = 0$$

$$\Rightarrow c = \frac{-2}{3} \Rightarrow a = \frac{1}{3},\ b = -\frac{1}{3}$$

Logo,

$$\int \frac{1}{x^3-1}\,dx = \int \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{3}\int \frac{1}{2}\frac{2x+1}{x^2+x+1}\,dx$$

$$-\frac{1}{3}\int \frac{3}{2}\cdot\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \qquad \frac{\sqrt{3}}{2}$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{6}\log(x^2+x+1)$$

$$-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{1}{3}\log\frac{|x-1|}{\sqrt{x^2+x+1}} - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

(m) $$\frac{x^3+2x}{x^4+4x^2+3} = \frac{x^3+2x}{(x^2+1)(x^2+3)} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+3}$$

$$= \frac{ax^3 + bx^2 + 3ax + 3b + cx^3 + dx^2 + cx + d}{(x^2+1)(x^2+3)}$$

$$\Rightarrow \begin{cases} a + c = 1 \Rightarrow c = 1 - a \\ b + d = 0 \Rightarrow d = -b \\ 3a + c = 2 \Rightarrow 3a + 1 - a = 2 \Rightarrow a = 1/2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow c = 1/2 \\ 3b + d = 0 \Rightarrow 3b - b = 0 \Rightarrow b = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow d = 0 \end{cases}$$

Assim,

$$\int \frac{x^3 + 2x}{x^4 + 4x^2 + 3}\,dx = \frac{1}{2}\int \frac{x}{x^2+1} + \frac{x}{x^2+3}\,dx$$

$$= \frac{1}{4}\left[\log(x^2+1) + \log(x^2+3)\right]$$

$$= \frac{1}{4}\log(x^4 + 4x^2 + 3)$$

(h) $\dfrac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2} = \dfrac{ax + b}{x^2 - 4x + 6} + \dfrac{cx + d}{(x^2 - 4x + 6)^2}$

$$= \frac{ax^3 + bx^2 - 4ax^2 - 4bx + 6ax + 6b + cx + d}{(x^2 - 4x + 6)^2}$$

$$\Rightarrow \begin{cases} a = 0 \\ b - 4a = 1 \Rightarrow b = 1 \\ -4b + 6a + c = -3 \Rightarrow c = 1 \\ 6b + d = 7 \Rightarrow d = 1 \end{cases}$$

Logo,

$$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx = \int \frac{1}{x^2-4x+6} + \frac{x+1}{(x^2-4x+6)^2}\,dx$$

$$= \int \frac{1}{(x-2)^2+(\sqrt{2})^2} + \frac{1}{2}\,\frac{2x-4}{(x^2-4x+6)^2} + \frac{3}{4}\cdot\frac{1}{\left[\left(\frac{x-2}{\sqrt{2}}\right)^2+1\right]^2}\,dx$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x-2}{\sqrt{2}}\right) - \frac{1}{2}\cdot\frac{1}{x^2-4x+6} + \frac{3}{4}\int \frac{dx}{\left[\left(\frac{x-2}{\sqrt{2}}\right)^2+1\right]^2}$$

Para calcular a integral, faça

$$\frac{x-2}{\sqrt{2}} = \tan u \Rightarrow dx = \sqrt{2}\sec^2 u\,du$$

$$\int \frac{dx}{\left[\left(\frac{x-2}{\sqrt{2}}\right)^2+1\right]^2} = \sqrt{2}\int \frac{\sec^2 u\,du}{\sec^4 u\,du} = \sqrt{2}\int \cos^2 u\,du$$

$$2\cos^2 u - 1 = \cos^2 u - \operatorname{sen}^2 u = \cos 2u$$

$$\Rightarrow \cos^2 u = \frac{\cos 2u + 1}{2}$$

$$= \frac{1}{\sqrt{2}}\int \cos 2u + 1\,du = \frac{1}{\sqrt{2}}(\operatorname{sen} u\cdot\cos u + u)$$

$$= \frac{1}{\sqrt{2}}\,\frac{\tan u}{1+\tan^2 u} + \frac{u}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\,\frac{\frac{x-2}{\sqrt{2}}}{1+\left(\frac{x-2}{\sqrt{2}}\right)^2} + \frac{1}{\sqrt{2}}\arctan\frac{x-2}{\sqrt{2}}$$

$$= \frac{x-2}{2+(x-2)^2} + \frac{1}{\sqrt{2}} \arctan \frac{x-2}{\sqrt{2}}$$

$$= \frac{x-2}{x^2-4x+6} + \frac{1}{\sqrt{2}} \arctan \frac{x-2}{\sqrt{2}}$$

Portanto

$$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx = \frac{1}{\sqrt{2}} \arctan \frac{x-2}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{x^2-4x+6}$$

$$+ \frac{3}{4} \frac{x-2}{x^2-4x+6} + \frac{3}{4} \cdot \frac{1}{\sqrt{2}} \arctan \frac{x-2}{\sqrt{2}}$$

$$= \frac{7}{4\sqrt{2}} \arctan \frac{x-2}{\sqrt{2}} + \frac{1}{4} \frac{3x-8}{x^2-4x+6}$$

(o) Repare que

$$\frac{d}{dx}\left(x^5+5x^3+5x\right) = 5x^4+5\cdot 3x^2+5$$
$$= 5\left(x^4+3x^2+1\right)$$

Logo,

$$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx = \frac{1}{5} \log\left|x^5+5x^3+5x\right|$$

Exercício 3. Faça uma substituição para expressar o integrando como uma função racional e então calcule a integral:

(a) $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$ (b) $\displaystyle\int \frac{dx}{x^2+x\sqrt{x}}$ (c) $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

(d) $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ (e) $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$ (f) $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

(g) $\displaystyle\int \frac{1}{1+\sqrt[3]{x}}\,dx$ (h) $\displaystyle\int \frac{dx}{(1+\sqrt{x})^2}$ (i) $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$

(j) $\displaystyle\int \frac{\sec^2 t}{\tan^2 t+2\tan t+2}\,dt$ (k) $\displaystyle\int \frac{dx}{1+e^x}$ (l) $\displaystyle\int \frac{\operatorname{sen} x}{\cos^2 x-3\cos x}\,dx$

(m) $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx$ (n) $\displaystyle\int \frac{\cosh t}{\operatorname{senh}^2 t+\operatorname{senh}^4 t}\,dt$

(a) $u=\sqrt{x-1} \Rightarrow du=\dfrac{dx}{2u} \Rightarrow dx=2u\,du$

$\Rightarrow x-1=u^2 \Rightarrow x=1+u^2$

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2\cancel{u}\,du}{(1+u^2)\cancel{u}} = 2\arctan u$$
$$= 2\arctan\sqrt{x-1}$$

(b) $u=\sqrt{x} \Rightarrow x=u^2 \Rightarrow dx=2u\,du$

$$\int \frac{dx}{x^2+x\sqrt{x}} = \int \frac{2u\,du}{u^4+u^3} = \int \frac{2\,du}{u^2(u+1)}$$

Temos

$$\frac{1}{u^2(u+1)} = \frac{a}{u}+\frac{b}{u^2}+\frac{c}{u+1}$$
$$= \frac{au^2+au+bu+b+cu^2}{u^2(u+1)}$$

$$\Rightarrow \begin{cases} a+c=0 \Rightarrow c=-a \\ a+b=0 \Rightarrow a=-b=-1 \Rightarrow c=1 \\ \quad b=1 \end{cases}$$

Logo,

$$\int \frac{dx}{x^2+x\sqrt{x}} = 2\int \frac{-1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\,du$$

$$= -2\log u - \frac{2}{u} + 2\log(u+1)$$

$$= 2\log\frac{\sqrt{x}+1}{\sqrt{x}} - \frac{2}{\sqrt{x}}$$

(c) $u = \sqrt[3]{x^2+1} \Rightarrow x^2+1 = u^3 \Rightarrow x = \sqrt{u^3-1}$

$$\Rightarrow dx = \frac{3u^2}{2\sqrt{u^3-1}}\,du$$

$$\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx = \int \frac{3\,(u^3-1)^{3/2}\cdot(u^3-1)^{-1/2}u^2\,du}{2u}$$

$$= \frac{3}{2}\int (u^3-1)\,u\,du = \frac{3}{2}\cdot\left(\frac{u^4}{4} - u\right)$$

$$= \frac{3}{2}\left[\frac{(x^2+1)^{4/3}}{4} - (x^2+1)^{1/3}\right]$$

(d) $u = x^{1/6} \Rightarrow x = u^6 \Rightarrow dx = 6u^5\,du$

$x^{1/2} = u^3, \quad u^{1/3} = u^2$

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 - u^2} = \int \frac{6u^3}{u-1}\,du$$

$$= 6\int \frac{u^3-1}{u-1} + \frac{1}{u-1}\,du$$

$$= 6(u^2+u+1) + \log|u-1|$$

$$= 6\left(x^{1/3} + x^{1/6} + 1\right) + \log|x^{1/6} - 1|$$

(e) $u = \sqrt{1+\sqrt{x}} \Rightarrow 1+\sqrt{x} = u^2 \Rightarrow x = (u^2-1)^2$

$dx = 4u(u^2-1)\,du$

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{4u^2(u^2-1)\,du}{(u^2-1)^2} = \int \frac{4u^2\,du}{u^2-1}$$

$$= \int \frac{4(u^2-1+1)}{u^2-1}\,du = \int 4 + \frac{4}{(u-1)(u+1)}\,du$$

$$= 4u + 2\int \frac{1}{u-1} - \frac{1}{u+1}\,du$$

$$= 4u + 2\log\left|\frac{u-1}{u+1}\right|$$

$$= 4\sqrt{1+\sqrt{x}} + 2\log\left|\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right|$$

(f) $u = \sqrt{x+3} \Rightarrow x = u^2 - 3 \Rightarrow dx = 2u\,du$

$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+u^2-3} = 2\int \frac{u\,du}{(u+3)(u-1)}$$

$$= \frac{1}{2}\int \frac{1}{u-1} - \frac{1}{u+3}\,du = \frac{1}{2}\log\left|\frac{u-1}{u+3}\right|$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x+3}-1}{\sqrt{x+3}+3}\right|$$

(g) $u = \sqrt[3]{x} \Rightarrow x = u^3 \Rightarrow dx = 3u^2\,du$

$$\int \frac{1}{1+\sqrt[3]{x}}\,dx = \int \frac{3u^2\,du}{1+u} = 3\int \frac{u^2-1+1}{u+1}\,du$$

$$= 3\int (u-1) + \frac{1}{u+1}\,du$$

$$= 3\left[\frac{u^2}{2} - u + \log|u+1|\right]$$

$$= \frac{3}{2}x^{2/3} - 3x^{1/3} + 3\log|x^{1/3}+1|$$

(h) $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\,du$

$$\int \frac{dx}{(1+\sqrt{x})^2} = \int \frac{2u\,du}{(1+u)^2} = \int \frac{2u+2-2}{u^2+2u+1}\,du$$

$$= \log\left[(u+1)^2\right] + \frac{2}{u+1}$$

$$= 2\log(\sqrt{x}+1) + \frac{2}{\sqrt{x}+1}$$

(i) $u = e^x \Rightarrow x = \log u \Rightarrow dx = \frac{du}{u}$

$$\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx = \int \frac{u^{\cancel{2}}\,du}{\cancel{u}(u^2+3u+2)}$$

$$= \int \frac{u\,du}{(u+1)(u+2)} = \int \frac{2}{u+2} - \frac{1}{u+1}\,du$$

$$= 2\log|u+2| - \log|u+1|$$

$$= 2\log(e^x+2) - \log(e^x+1)$$

(j) $u = \tan t \Rightarrow du = \sec^2 t\,dt$

$$\int \frac{\sec^2 t}{\tan^2 t + 2\tan t + 2}\,dt = \int \frac{du}{u^2+2u+2}$$

$$= \int \frac{du}{(u+1)^2+1} = \arctan(u+1)$$

$$= \arctan(1+\tan t)$$

(k) $u = e^x \Rightarrow x = \log u \Rightarrow dx = \frac{du}{u}$

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u(u+1)} = \int \frac{1}{u} - \frac{1}{u+1}\, du$$

$$= \log\left|\frac{u}{u+1}\right| = \log\left(\frac{e^x}{e^x+1}\right)$$

$$= x - \log(e^x+1)$$

(l) $u = \cos x \Rightarrow du = -\operatorname{sen} x\, dx$

$$\int \frac{\operatorname{sen} x\, dx}{\cos^2 x - 3\cos x} = -\int \frac{du}{u^2 - 3u} = -\int \frac{du}{u(u-3)}$$

$$= \frac{1}{3}\int \frac{1}{u} - \frac{1}{u-3}\, du = \frac{1}{3} \log\left|\frac{u}{u-3}\right|$$

$$= \frac{1}{3} \log\left|\frac{\cos x}{\cos x - 3}\right|$$

(m) $u = e^x \Rightarrow du = e^x dx$

$$\int \frac{e^x\, dx}{(e^x-2)(e^{2x}+1)} = \int \frac{du}{(u-2)(u^2+1)}$$

$$= \frac{1}{2}\int \frac{1}{u-2} - \frac{u}{u^2+1}\, du$$

$$= \frac{1}{2}\log|u-2| - \frac{1}{4}\log(u^2+1)$$

$$= \frac{1}{2}\log\frac{(e^x-2)^2}{e^{2x}+1}$$

(n) $u = \operatorname{senh} t \Rightarrow du = \cosh t\, dt$

$$\int \frac{\cosh t \; dt}{\operatorname{senh}^2 t + \operatorname{senh}^4 t} = \int \frac{du}{u^2(u^2+1)}$$

$$= \int \frac{1}{u^2} - \frac{1}{u^2+1}\, du = -\frac{1}{u} - \arctan u$$

$$= \frac{-1}{\operatorname{senh} t} - \arctan(\operatorname{senh} t)$$

Exercício 4. Calcule:

(a) $\int \log(x^2 - x + 2)\,dx$ (b) $\int x \arctan x\,dx$

(a) $\int \log(x^2-x+2)\,dx = x\log(x^2-x+2) - \int \frac{x(2x-1)}{x^2-x+2}\,dx$

Repare que, dividindo,

$$\begin{array}{r|l} 2x^2 - x & x^2-x+2 \\ \hline -2x^2+2x-4 & 2 \\ \hline x-4 & \end{array}$$

Logo,

$$\frac{2x^2-x}{x^2-x+2} = 2 + \frac{x-4}{x^2-x+2}$$

Assim,

$$\int \frac{2x^2-x}{x^2-x+2}\,dx = \int 2 + \frac{1}{2}\frac{2x-1}{x^2-x+2} + \frac{9}{2}\frac{1}{x^2-x+2}\,dx$$

$$= 2x + \frac{1}{2}\log(x^2-x+2) + \frac{9}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}\,dx$$

$$= 2x + \frac{1}{2}\log(x^2-x+2) + \frac{9}{2}\cdot\frac{2}{\sqrt{7}}\arctan\left(\frac{x-1/2}{\sqrt{7}/2}\right)$$

$=$

Assim,

$$\int \log(x^2-x+2)\,dx = x\log(x^2-x+2) - \left[2x + \frac{1}{2}\log(x^2-x+2) + \frac{9}{\sqrt{7}}\arctan\left(\frac{2x-1}{\sqrt{7}}\right)\right]$$

(b)

$$\int x\arctan x\,dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x$$

$$= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x$$

Exercício 5. Usando a substituição $t = \tan(x/2)$, calcule:

(a) $\displaystyle\int \frac{dx}{1-\cos x}$ (b) $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1+\operatorname{sen} x - \cos x}\,dx$

(c) $\displaystyle\int \frac{1}{3\operatorname{sen} x - 4\cos x}\,dx$ (d) $\displaystyle\int_{0}^{\pi/2} \frac{\operatorname{sen} 2x}{2+\cos x}\,dx$

Em todos os itens, usaremos o seguinte:

- $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = (1+t^2)\,dx$

$$\Rightarrow dx = \frac{2}{1+t^2}\,dt$$

- $\operatorname{sen} x = 2\operatorname{sen}\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan x/2}{\sec^2 x/2} = \frac{2t}{1+t^2}$

- $\cos x = \cos^2\frac{x}{2} - \operatorname{sen}^2\frac{x}{2} = \frac{1-\tan^2 x/2}{\sec^2 x/2} = \frac{1-t^2}{1+t^2}$

(a) $$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-t^2}{1+t^2}}\,\frac{2}{(1+t^2)}\,dt$$

$$= \int \frac{2\,dt}{\cancel{1}+t^2-\cancel{1}+t^2} = \int t^{-2}\,dt = -\frac{1}{t} = -\cot\frac{x}{2}$$

(b) $$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\operatorname{sen} x-\cos x} = \int_{1/\sqrt{3}}^{1} \frac{1}{1+\frac{2t-1+t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$$

$$= \int_{1/\sqrt{3}}^{1} \frac{2}{\cancel{1}+t^2+2t-\cancel{1}+t^2}\,dt = \int_{1/\sqrt{3}}^{1} \frac{1}{t(1+t)}\,dt$$

$$= \int_{1/\sqrt{3}}^{1} \frac{1}{t} - \frac{1}{1+t}\, dt = \log \frac{t}{1+t} \Big|_{1/\sqrt{3}}^{1}$$

$$= \log \frac{1}{2} - \log\left(\frac{1/\cancel{\sqrt{3}}}{(\sqrt{3}+1)/\cancel{\sqrt{3}}}\right)$$

$$= \log \frac{\sqrt{3}+1}{2}$$

(c) $$\int \frac{1}{3\operatorname{sen}x - 4\cos x}\, dx = \int \frac{1}{\frac{6t - 4 + 4t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int \frac{dt}{2t^2+3t-2}$$

O denominador tem raízes

$$t = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4} = -2 \text{ e } \frac{1}{2}$$

Logo,

$$\int \frac{1}{3\operatorname{sen}x - 4\cos x}\, dx = \int \frac{dt}{(t+2)(2t-1)}$$

$$= \int \frac{2}{2t-1} - \frac{1}{t+2}\, dt = \log\left|\frac{2t-1}{t+2}\right|$$

$$= \log \left| \frac{2\tan\frac{x}{2} - 1}{\tan\frac{x}{2} + 2} \right|$$

(d) $$\int_0^{\pi/2} \frac{\operatorname{sen} 2x}{2+\cos x}\,dx = \int_0^{\pi/2} \frac{2\operatorname{sen} x \cos x}{2+\cos x}\,dx$$

$$= \int_0^1 \frac{\frac{4t(1-t^2)}{(1+t^2)^2}}{2 + \frac{1-t^2}{(1+t^2)}} \cdot \frac{2t}{1+t^2}\,dt$$

$$= \int_0^1 \frac{8t^2(1-t^2)\;dt}{(1+t^2)^2(2+2t^2+1-t^2)} = \int_0^1 \frac{8t^2(1-t^2)}{(1+t^2)^2(3+t^2)}\,dt$$

Temos

$$\frac{8t^2 - 8t^4}{(1+t^2)^2(3+t^2)} = \frac{at+b}{1+t^2} + \frac{ct+d}{(1+t^2)^2} + \frac{et+f}{3+t^2}$$

$$= \left[(at^3+bt^2+at+b)(t^2+3) + ct^3 + dt^2 + 3ct + 3d \right.$$
$$\left. + (et^3+ft^2+et+f)(t^2+1)\right] / (1+t^2)^2(3+t^2)$$

$$= \left[at^5 + bt^4 + at^3 + bt^2 + 3at^3 + 3bt^2 + 3at + 3b \right.$$
$$+ ct^3 + dt^2 + 3ct + 3d + et^5 + ft^4 + et^4 + ft^2 +$$
$$\left. et^3 + ft^2 + et + f\right] / (1+t^2)^2(3+t^2)$$

$$\Rightarrow\begin{cases} a+e=0 \Rightarrow e=-a = 0 \\ b+f+e=-8 \Rightarrow f = a-b-8 = -b-8 = -24 \\ 4a+c+e=0 \Rightarrow c=-3a = 0 \\ 4b+d+2f=8 \Rightarrow 2b+d+2a=24 \Rightarrow d=24-2a-2b \quad d=24-2b = -8 \\ 3a+3c+e=0 \Rightarrow -7a=0 \Rightarrow a=0 \\ 3b+3d+f=0 \Rightarrow 2b+3d=8 \end{cases}$$

$$\Rightarrow 2b-6b+72=8$$

$$\Rightarrow 4b=64 \Rightarrow b=16$$

Ou seja,

$$\frac{8t^2-8t^4}{(1+t^2)^2(3+t^2)} = \frac{16}{1+t^2} - \frac{8}{(1+t^2)^2} - \frac{24}{3+t^2}$$

Assim,

$$\int_0^{\pi/2} \frac{\operatorname{sen} 2x}{2+\cos x}\,dx = \int_0^1 \frac{16}{1+t^2} - \frac{8}{(1+t^2)^2} - \frac{24}{3+t^2}\,dt$$

$$= \left(16 \arctan t - \frac{24}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}}\right)\Bigg|_0^1 - 8\int_0^1 \frac{dt}{(1+t^2)^2}$$

$$= 16\cdot\frac{\pi}{4} - \frac{24}{\sqrt{3}}\cdot\frac{\pi}{6} - 8\int_0^1 \frac{dt}{(1+t^2)^2}$$

$$= \left(4 - \frac{4}{\sqrt{3}}\right)\pi - 8\int_0^1 \frac{dt}{(1+t^2)^2}$$

Para a calcular a última integral, faça

$$t = \tan\theta \Rightarrow dt = \sec^2\theta \, d\theta$$

$$\int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = \int_0^{\pi/4} \cos^2\theta \, d\theta$$

$$2\cos^2\theta - 1 = \cos^2\theta - \operatorname{sen}^2\theta = \cos 2\theta$$

$$\Rightarrow \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$= \int_0^{\pi/4} \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{\theta}{2} + \frac{1}{4} \operatorname{sen} 2\theta \Big|_0^{\pi/4}$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

Logo,

$$\int_0^{\pi/2} \frac{\operatorname{sen} 2x}{2 + \cos x} \, dx = \left(4 - \frac{4}{\sqrt{3}}\right)\pi - 8\left(\frac{\pi}{8} + \frac{1}{4}\right)$$

$$= 3\pi - \frac{4\pi}{\sqrt{3}} - 2$$

Exercício 6. Calcule $\displaystyle\int \frac{dx}{x^4+1}$

Uma ideia é tentar fatorar

$$x^4+1 = [(x^2+1)-a][(x^2+1)+a]$$

$$= x^4+2x^2+1-a^2$$

$$\Rightarrow a^2 = 2x^2 \Rightarrow \quad a = \sqrt{2}x$$

Temos

$$\frac{1}{x^4+1} = \frac{1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}$$

$$= \frac{ax+b}{x^2-\sqrt{2}x+1} + \frac{cx+d}{x^2+\sqrt{2}x+1}$$

$$= [ax^3+\sqrt{2}ax^2+ax+bx^2+b\sqrt{2}x+b$$

$$+ cx^3-\sqrt{2}cx^2+cx+dx^2-d\sqrt{2}x+d]/(x^4+1)$$

$$\Rightarrow \begin{cases} a+c = 0 \Rightarrow c=-a \\ \sqrt{2}a+b-\sqrt{2}c+d = 0 \\ a+b\sqrt{2}+c-d\sqrt{2} = 0 \\ b+d = 1 \Rightarrow d = 1-b \end{cases}$$

$$\Rightarrow 2\sqrt{2}a+\cancel{b}+1-\cancel{b} = 0 \Rightarrow a = -\frac{1}{2\sqrt{2}}, \quad c = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow b\sqrt{2} = d\sqrt{2} \Rightarrow b = d$$

$$\Rightarrow b = 1-b \Rightarrow b = d = 1/2$$

Assim,

$$\int \frac{dx}{x^4+1} = \int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2+\sqrt{2}x+1} - \frac{\frac{1}{2\sqrt{2}}x - \frac{1}{2}}{x^2-\sqrt{2}x+1}\, dx$$

$$= \int \left[\frac{1}{4\sqrt{2}} \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{1}{4} \frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} \right.$$

$$\left. - \frac{1}{4\sqrt{2}} \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} + \frac{1}{4} \frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} \right] dx$$

Logo,

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}} \log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)$$

$$+ \frac{1}{4}\sqrt{2} \arctan\left(\frac{x+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) + \frac{1}{4}\sqrt{2} \arctan\left(\frac{x-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)$$

$$= \frac{1}{4\sqrt{2}} \log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{4} \arctan(\sqrt{2}x+1)$$

$$+ \frac{\sqrt{2}}{4} \arctan(\sqrt{2}x-1)$$

É possível fazer uma simplificação final:

$$\tan(a+b) = \frac{\operatorname{sen} a \cos b + \operatorname{sen} b \cos a}{\cos a \cos b - \operatorname{sen} b \operatorname{sen} b}$$

$$= \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$$

$$\Rightarrow a+b = \arctan\left[\frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}\right]$$

Assim,

$$\arctan \alpha + \arctan \beta = \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

Para $\alpha = \sqrt{2}x + 1$, $\beta = \sqrt{2}x - 1$, vem

$$\frac{\alpha+\beta}{1-\alpha\beta} = \frac{2\sqrt{2}x}{1-(2x^2-1)} = \frac{\sqrt{2}x}{1-x^2}$$

Logo,

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}} \log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)$$

$$+ \frac{\sqrt{2}}{4} \arctan\left(\frac{\sqrt{2}x}{1-x^2}\right)$$

Exercício 7. Se f for uma função quadrática tal que $f(0) = 1$ e

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx$$

for uma função racional, encontre o valor de $f'(0)$.

$$f(x) = \alpha x^2 + \beta x + \gamma \qquad f(0) = \gamma = 1$$

$$\frac{f(x)}{x^2(x+1)^3} = \frac{\alpha x^2 + \beta x + 1}{x^2(x+1)^3} =$$

$$= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2} + \frac{e}{(x+1)^3}$$

$$= \Big[ax^4 + 3ax^3 + 3ax^2 + ax + bx^3 + 3bx^2 + 3bx$$
$$+ b + cx^4 + 2cx^3 + cx^2 + dx^3 + dx^2 + ex^2\Big] / \left[x^2(x+1)^3\right]$$

$$\Rightarrow \begin{cases} a + c = 0 \Rightarrow c = -a = 3-\beta \\ 3a + b + 2c + d = 0 \Rightarrow a + d = -1 \Rightarrow d = -1 - a \\ \qquad \Rightarrow d = 2-\beta \\ 3a + 3b + c + d + e = \alpha \\ a + 3b = \beta \Rightarrow a = \beta - 3 \\ b = 1 \end{cases}$$

$$\Rightarrow 3\beta - 9 + 3 + 3 - \beta + 2 - \beta + e = \alpha$$

$$\Rightarrow \beta - 1 + e = \alpha \Rightarrow e = 1 + \alpha - \beta$$

Assim,

$$a = \beta - 3 \qquad b = 1 \qquad c = 3 - \beta$$

$$d = 2 - \beta \qquad e = 1 + \alpha - \beta$$

Como a integral é racional, de

$$\int \frac{\alpha x^2+\beta x+1}{x^2(x+1)^3}\,dx = \int \frac{\beta-3}{x} + \frac{1}{x^2} + \frac{3-\beta}{x+1}$$

$$+ \frac{2-\beta}{(x+1)^2} + \frac{1+\alpha-\beta}{(x+1)^3}\,dx$$

$$= (\beta-3)\log x - \frac{1}{x} + (3-\beta)\log(x+1)$$

$$- \frac{(2-\beta)}{x+1} - \frac{(1+\alpha-\beta)}{2(x+1)^2}$$

segue que os termos no logaritmo têm que ser nulos. Logo, $\beta = 3$.

Mas

$$f'(x) = 2\alpha x + \beta \Rightarrow f'(0) = \beta = 3.$$

Exercício 8. Se $a \neq 0$ e $n \in \mathbb{N}$, encontre a decomposição em frações parciais de

$$f(x) = \frac{1}{x^n(x-a)}$$

Temos que determinar $a_1, a_2, \dots, a_n, b$ tais que

$$\frac{1}{x^n(x-a)} = \sum_{k=1}^{n} \frac{a_k}{x^k} + \frac{b}{x-a}$$

$$= \sum_{k=1}^{n} \frac{a_k x^{n-k}}{x^n} + \frac{b}{x-a} \qquad \begin{matrix} n-1 \\ -a \cdot a_1 \end{matrix} \qquad a_2 - a \cdot a_1 x^{n-1}$$

$$= \frac{\sum_{k=1}^{n} a_k x^{n+1-k} - \sum_{k=1}^{n} a \cdot a_k x^{n-k} + b x^n}{x^n(x-a)}$$

$$= \frac{(a_1+b)x^n + \sum_{k=1}^{n-1}(a_{k+1} - a \cdot a_k)x^{n-k} - a \cdot a_n}{x^n(x-a)}$$

Logo,

$$-a \cdot a_n = 1 \Rightarrow a_n = -1/a$$

$$a_n - a \cdot a_{n-1} = 0 \Rightarrow a_{n-1} = \frac{a_n}{a} = \frac{-1}{a^2}$$

$$a_{n-1} - a \cdot a_{n-2} = 0 \Rightarrow a_{n-2} = \frac{-1}{a^3}$$

Se $a_{n-k} = -\frac{1}{a^{k+1}}$ $(k < n-1)$, então

$$a_{n-k} - a \cdot a_{n-(k+1)} = 0$$

$$\Rightarrow a_{n-(k+1)} = \frac{a_{n-k}}{a} = -\frac{1}{a^{k+2}}$$

Assim, prova-se por indução que

$$a_{n-k} = -\frac{1}{a^{k+1}} \quad (0 \leq k \leq n-1)$$

Por fim,

$$a_1 + b = 0 \Rightarrow b = -a_1 = \frac{1}{a^n}$$

Logo,

$$\frac{1}{x^n(x-a)} = \frac{a^{-n}}{x-a} - \sum_{k=1}^{n} \frac{a^{-(n+1-k)}}{x^k}$$

Exercício 9. Calcule $\int \frac{dx}{x^7 - x}$

$$\int \frac{dx}{x^7-x} = \int \frac{dx}{x(x^6-1)} = \int \frac{x^5}{x^6-1} - \frac{1}{x}\, dx$$

$$= \frac{1}{6}\log|x^6-1| - \log|x|$$

$$= \frac{1}{6}\log\left|\frac{x^6-1}{x^6}\right| = \frac{1}{6}\log|1-x^{-6}|$$

Exercício 10. Calcule $\int \sqrt{\tan x}\, dx$

$u = \sqrt{\tan x} \Rightarrow x = \arctan u^2 \Rightarrow dx = \dfrac{2u\,du}{1+u^4}$

$$\int \sqrt{\tan x}\, dx = \int \frac{2u^2}{1+u^4}\, du$$

Queremos achar a tal que

$$1+u^4 = (1+u^2+a)(1+u^2-a)$$
$$= (1+u^2)^2 - a^2 = 1+u^4+2u^2-a^2$$

$$\Rightarrow a = \sqrt{2}u$$

Assim,

$$\frac{2u^2}{1+u^4} = \frac{2u^2}{(u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)}$$

$$= \frac{au+b}{u^2-\sqrt{2}u+1} + \frac{cu+d}{u^2+\sqrt{2}u+1}$$

$$= [au^3+bu^2+a\sqrt{2}u^2+b\sqrt{2}u+au+b$$
$$+ cu^3+du^2-c\sqrt{2}u^2-d\sqrt{2}u+cu+d]/(1+u^4)$$

$$\Rightarrow \begin{cases} a+c=0 \Rightarrow c=-a \\ b+a\sqrt{2}+d-c\sqrt{2}=2 \\ b\sqrt{2}+a-d\sqrt{2}+c=0 \Rightarrow b=d \\ b+d=0 \Rightarrow d=-b \Rightarrow b=d=0 \end{cases}$$

$$\Rightarrow 2a\sqrt{2}=2 \Rightarrow a=\frac{1}{\sqrt{2}},\ c=-\frac{1}{\sqrt{2}}$$

Logo,

$$\int \frac{2u^2}{1+u^4}\,du = \int\left[\frac{1}{2\sqrt{2}}\,\frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1} + \frac{1}{2}\,\frac{1}{\left(u-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right.$$

$$\left. - \frac{1}{2\sqrt{2}}\,\frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1} + \frac{1}{2}\,\frac{1}{\left(u+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right] du$$

$$= \frac{1}{2\sqrt{2}}\log\left(\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right) + \frac{1}{2}\sqrt{2}\arctan\left(\frac{u-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)$$

$$+ \frac{1}{2}\sqrt{2}\arctan\left(\frac{u+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)$$

$$= \frac{1}{2\sqrt{2}}\log\left(\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right) + \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}u-1\right)$$

$$+ \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}u+1\right)$$

Como $\arctan\alpha + \arctan\beta = \arctan\dfrac{\alpha+\beta}{1-\alpha\beta}$,

$$\int \frac{2u^2}{1+u^4}\,du = \frac{1}{2\sqrt{2}}\log\left(\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right)$$

$$+ \frac{1}{\sqrt{2}}\arctan\frac{2\sqrt{2}u}{1-(2u^2-1)}$$

$$= \frac{1}{2\sqrt{2}} \log\left(\frac{u^2 - \sqrt{2}u + 1}{u^2 + \sqrt{2}u + 1}\right) + \frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{2}u}{1 - u^2}\right)$$

Portanto, como $u = \sqrt{\tan x}$,

$$\int \sqrt{\tan x}\, dx = \frac{1}{2\sqrt{2}} \log\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right)$$

$$+ \frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{2\tan x}}{1 - \tan x}\right)$$

Exercício 11. Calcule $\int \frac{dx}{x^6+1}$

Temos que

$$x^6+1 = (x^2)^3+1 = (x^2+1)\left((x^2)^2-x^2+1\right)$$
$$= (x^2+1)(x^4-x^2+1)$$

Assim,

$$\frac{1}{x^6+1} = \frac{1}{2}\,\frac{2}{x^6+1} = \frac{1}{2}\,\frac{x^4+1-(x^4-1)}{(x^2+1)(x^4-x^2+1)}$$

$$= \frac{1}{2}\,\frac{(x^4-x^2+1)+x^2}{(x^2+1)(x^4-x^2+1)} - \frac{1}{2}\,\frac{x^4-1}{(x^2+1)(x^4-x^2+1)}$$

$$= \frac{1}{2}\,\frac{1}{x^2+1} + \frac{1}{2}\,\frac{x^2}{x^6+1} - \frac{1}{2}\,\frac{(x^2-1)\cancel{(x^2+1)}}{\cancel{(x^2+1)}(x^4-x^2+1)}$$

$$= \frac{1}{2}\,\frac{1}{x^2+1} + \frac{1}{6}\,\frac{3x^2}{(x^3)^2+1} - \frac{1}{2}\,\frac{x^2-1}{x^4-x^2+1}$$

Logo,

$$\int \frac{dx}{x^6+1} = \frac{1}{2}\arctan x + \frac{1}{6}\arctan x^3$$
$$-\frac{1}{2}\int \frac{x^2-1}{x^4-x^2+1}\,dx$$

Resta avaliar a última integral:

$$\int \frac{x^2-1}{x^4-x^2+1}\,dx = \int \frac{1-x^{-2}}{x^2-1+x^{-2}}\,dx$$

$$= \int \frac{1 - x^{-2}}{(x + x^{-1})^2 - 3}\, dx \qquad \left\{ \begin{array}{l} u = x + x^{-1} \\ \Rightarrow du = (1 - x^{-2})\, dx \end{array} \right.$$

$$= \int \frac{du}{u^2 - 3} = \int \frac{du}{(u - \sqrt{3})(u + \sqrt{3})}$$

$$= \frac{1}{2\sqrt{3}} \int \frac{1}{u - \sqrt{3}} - \frac{1}{u + \sqrt{3}}\, du$$

$$= \frac{1}{2\sqrt{3}} \log\left| \frac{u - \sqrt{3}}{u + \sqrt{3}} \right| = \frac{1}{2\sqrt{3}} \log\left| \frac{x + x^{-1} - \sqrt{3}}{x + x^{-1} + \sqrt{3}} \right|$$

$$= \frac{1}{2\sqrt{3}} \log\left(\frac{x^2 - \sqrt{3}x + 1}{x^2 + \sqrt{3}x + 1} \right)$$

Portanto,

$$\int \frac{dx}{x^6 + 1} = \frac{1}{2} \arctan x + \frac{1}{6} \arctan x^3$$

$$+ \frac{1}{4\sqrt{3}} \log\left(\frac{x^2 + \sqrt{3}x + 1}{x^2 - \sqrt{3}x + 1} \right)$$

Exercício 1. Explique por que cada uma das seguintes integrais é imprópria:

(a) $\displaystyle\int_1^2 \frac{x}{x-1}\,dx$ (b) $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$ (c) $\displaystyle\int_0^{\infty} \frac{1}{1+x^3}\,dx$ (d) $\displaystyle\int_0^{\pi/4} \cot x\,dx$

(a) $\dfrac{x}{x-1} \xrightarrow[x \to 1_+]{} \infty$

(b) Limite infinito de integração

(c) Limite infinito de integração

(d) $\cot x \xrightarrow[x \to 0_+]{} \infty$

Exercício 2. Determine se as seguintes integrais são convergentes ou divergentes e calcule seu valor quando convergirem:

(a) $\displaystyle\int_3^\infty \frac{dx}{(x-2)^{3/2}}$ (b) $\displaystyle\int_{-\infty}^0 \frac{dx}{3-4x}$ (c) $\displaystyle\int_2^\infty e^{-5p}\,dp$

(d) $\displaystyle\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx$ (e) $\displaystyle\int_0^\infty \operatorname{sen}^2\alpha\,d\alpha$ (f) $\displaystyle\int_1^\infty \frac{x+1}{x^2+2x}\,dx$

(g) $\displaystyle\int_e^\infty \frac{1}{x(\log x)^3}\,dx$ (h) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx$ (i) $\displaystyle\int_{-\infty}^0 \frac{z}{z^4+4}\,dz$

(j) $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy$ (k) $\displaystyle\int_0^1 \frac{dx}{x}$ (l) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

(m) $\displaystyle\int_{-\infty}^0 2^r\,dr$ (n) $\displaystyle\int_1^\infty \frac{dx}{(2x+1)^3}$ (o) $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$

(p) $\displaystyle\int_0^\infty \operatorname{sen}\theta\, e^{\cos\theta}\,d\theta$

(a) $$\int_3^\infty \frac{dx}{(x-2)^{3/2}} = \frac{-2}{(x-2)^{1/2}}\Big|_3^\infty = 2$$

(b) $$\int_{-\infty}^0 \frac{dx}{3-4x} \underset{u=-x}{=} \int_0^\infty \frac{du}{4u+3} = \frac{1}{4}\log(4u+3)\Big|_0^\infty = \infty$$

(c) $$\int_2^\infty e^{-5p}\,dp = -\frac{1}{5}e^{-5p}\Big|_2^\infty = \frac{e^{-10}}{2}$$

(d) $$\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\sqrt{1+x^3}\,\Big|_0^\infty = \infty$$

(e) $$\int_0^\infty \operatorname{sen}^2\alpha\,d\alpha \geq \sum_{k=1}^{\infty} \int_{2k\pi+\frac{\pi}{4}}^{2k\pi+\frac{\pi}{2}} \operatorname{sen}^2\alpha\,d\alpha$$

$$\geq \sum_{k=1}^{\infty} \frac{1}{2}\cdot\frac{\pi}{4} = \infty$$

(f) $\displaystyle\int_1^{\infty} \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\log(x^2+2x)\Big|_1^{\infty} = \infty$

(g) $\displaystyle\int_e^{\infty} \frac{1}{x(\log x)^3}\,dx = -\frac{1}{2}(\log x)^{-2}\Big|_e^{\infty} = \frac{1}{2}$

(h) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\int_0^{\infty} \frac{x^2}{9+x^6}\,dx$

$$= \frac{2}{9}\int_0^{\infty} \frac{x^2}{1+\left(\frac{x^3}{3}\right)^2}\,dx \quad \left\{\begin{array}{l} u = \frac{x^3}{3} \\ du = x^2\,dx \end{array}\right.$$

$$= \frac{2}{9}\int_0^{\infty} \frac{du}{1+u^2} = \frac{2}{9}\arctan u\Big|_0^{\infty}$$

$$= \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}$$

(i) $\displaystyle\int_{-\infty}^{0} \frac{z}{z^4+4}\,dz = \frac{1}{4}\int_{-\infty}^{0} \frac{z}{1+\left(\frac{z^2}{2}\right)^2}\,dz \quad \left\{\begin{array}{l} u = z^2/2 \\ du = z\,dz \end{array}\right.$

$$= \frac{1}{4}\int_{-\infty}^{0} \frac{du}{1+u^2} = \frac{1}{4}\arctan u\Big|_{-\infty}^{0} = \frac{\pi}{8}$$

(j) $\displaystyle\int_0^{\infty} e^{-\sqrt{y}}\,dy \quad \left\{\begin{array}{l} u = \sqrt{y} \\ du = \frac{1}{2u}\,dx \Rightarrow dx = 2u\,du \end{array}\right.$

$$= \int_0^\infty 2u e^{-u} du = -2u e^{-u} \Big|_0^\infty + 2\int_0^\infty e^{-u} du$$

$$= -2e^{-u} \Big|_0^\infty = 2$$

(k) $\displaystyle\int_0^1 \frac{dx}{x} = \log x \Big|_0^1 = \infty$

(l) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}} dx = \frac{4}{3}(1+x)^{3/4} \Big|_0^\infty = \infty$

(m) $\displaystyle\int_{-\infty}^0 2^r dr = \int_{-\infty}^0 e^{r\log 2} dr = \frac{2^r}{\log 2} \Big|_{-\infty}^0$

$$= \frac{1}{\log 2}$$

(n) $\displaystyle\int_1^\infty \frac{dx}{(2x+1)^3} = \frac{-1}{4}(2x+1)^{-2} \Big|_1^\infty = \frac{1}{36}$

(o) $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2} dx = e^{-1/x} \Big|_1^\infty = \frac{1}{e}$

(p) $\int_0^\infty \operatorname{sen}\theta \, e^{\cos\theta} \, d\theta = -e^{\cos\theta}\Big|_0^\infty$,

que não converge.

Exercício 3. Determine se as seguintes integrais são convergentes ou divergentes e calcule seu valor quando convergirem:

(a) $\displaystyle\int_0^\infty \frac{dz}{z^2+3z+2}$ (b) $\displaystyle\int_{-\infty}^\infty x^3e^{-x^4}\,dx$ (c) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\,dx$

(d) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^2}$ (e) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$ (f) $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx$

(g) $\displaystyle\int_0^1 \frac{3}{x^5}\,dx$ (h) $\displaystyle\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx$ (i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

(j) $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx$ (k) $\displaystyle\int_{-\pi/2}^\pi \operatorname{cossec} x\,dx$ (l) $\displaystyle\int_0^{\pi/2} \tan^2\theta\,d\theta$

(m) $\displaystyle\int_0^1 r\log r\,dr$ (n) $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$ (o) $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\operatorname{sen}\theta}}\,d\theta$

(a) $$\int_0^\infty \frac{dz}{(z+1)(z+2)} = \lim_{b\to\infty}\int_0^b \frac{1}{z+1}-\frac{1}{z+2}\,dz$$

$$= \lim_{b\to\infty} \log\left(\frac{z+1}{z+2}\right)\Big|_0^b = \log 2 + \lim_{b\to\infty}\log\frac{b+1}{b+2}$$

$$= \log 2.$$

(b) $$\int_{-\infty}^\infty x^3e^{-x^4}dx = \int_{-\infty}^0 x^3e^{-x^4}dx + \int_0^\infty x^3e^{-x^4}dx$$

O integrando é ímpar, então se a integral converge, ela tem que dar zero.

$$\int_0^\infty x^3e^{-x^4}dx = -\frac{1}{4}e^{-x^4}\Big|_0^\infty = \frac{1}{4}$$

$$\Rightarrow \int_{-\infty}^\infty x^3e^{-x^4}dx = 0.$$

(c) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\,dx = -\frac{1}{x}\log x\Big|_1^\infty + \int_1^\infty \frac{dx}{x^2}$

$\displaystyle = -\frac{1}{x}\Big|_1^\infty = 1$

(d) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^2} = -\frac{1}{\log x}\Big|_e^\infty = 1$

(e) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$ $\left\{\begin{array}{l} u=\sqrt{x} \\ \Rightarrow du = \frac{dx}{2u} \Rightarrow dx = 2u\,du \end{array}\right.$

$\displaystyle = \int_1^\infty \frac{2u\,du}{u+u^3} = \int_1^\infty \frac{2\,du}{1+u^2} = 2\arctan u\Big|_1^\infty$

$\displaystyle = 2\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{2}$

(f) $\displaystyle\int_0^5 \frac{dx}{\sqrt[3]{5-x}} = -\frac{3}{2}(5-x)^{2/3}\Big|_0^5 = \frac{3\sqrt[3]{25}}{2}$

(g) $\displaystyle\int_0^1 \frac{3}{x^5}\,dx = -\frac{3}{4}x^{-4}\Big|_0^1 = \infty$

(h) $\displaystyle\int_0^\infty \frac{x\cdot\arctan x}{(1+x^2)^2}\,dx = -\frac{1}{2}\,\frac{\arctan x}{1+x^2}\Big|_0^\infty$

$$+\frac{1}{2}\int_0^\infty \frac{1}{(1+x^2)^2}dx = \frac{1}{2}\int_0^\infty \frac{1}{(1+x^2)^2}dx$$

Note que

$$\int \frac{dx}{(1+x^2)} = \frac{x}{1+x^2} + \int \frac{2x^2}{(1+x^2)^2}dx$$

$$= \frac{x}{1+x^2} + 2\int \frac{1+x^2-1}{(1+x^2)^2}dx$$

$$= \frac{x}{1+x^2} + 2\int \frac{dx}{1+x^2} - 2\int \frac{dx}{(1+x^2)^2}$$

$$\Rightarrow \int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\arctan x$$

Logo,

$$\int_0^\infty \frac{x\cdot \arctan x}{(1+x^2)^2}dx = \frac{1}{4}\left[\frac{x}{1+x^2} + \arctan x\right]_0^\infty$$

$$= \frac{\pi}{8}$$

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_0^1 = \frac{\pi}{2}$

(j) A função é descontínua em $x=0$. Daí,

$$\int_{-2}^{3} \frac{dx}{x^4} = \int_{-2}^{0} \frac{dx}{x^4} + \int_{0}^{3} \frac{dx}{x^4}$$

No entanto, as integrais divergem:

$$\int_{-2}^{0} \frac{dx}{x^4} = \frac{-1}{3} x^{-3} \Big|_{-2}^{0} = \infty$$

$$\int_{0}^{3} \frac{dx}{x^4} = \frac{-1}{3} x^{-3} \Big|_{0}^{3} = \infty$$

(k) A função é descontínua em $x=0$. Daí,

$$\int_{-\pi/2}^{\pi} \operatorname{cossec} x \, dx = \int_{-\pi/2}^{0} \operatorname{cossec} x \, dx + \int_{0}^{\pi} \operatorname{cossec} x \, dx$$

Temos que

$$\int \operatorname{cossec} x \, dx = \int \frac{\operatorname{cossec} x(\operatorname{cossec} x + \cot x)}{\operatorname{cossec} x + \cot x} \, dx$$

$$= \int \frac{\operatorname{cossec} x \cot x + \operatorname{cossec}^2 x}{\operatorname{cossec} x + \cot x} \, dx$$

$$= -\log|\operatorname{cossec} x + \cot x|$$

Daí,

$$\int_{-\pi/2}^{0} \operatorname{cossec} x \, dx = -\log|\operatorname{cossec} x + \cot x|\Big|_{-\pi/2}^{0} = -\infty$$

$$\int_{0}^{\pi} \operatorname{cossec} x \, dx = -\log|\operatorname{cossec} x + \cot x|\Big|_{0}^{\pi} = \infty,$$

e então a integral diverge.

(e) $$\int_0^{\pi/2} \tan^2\theta \, d\theta = \int_0^{\pi/2} \frac{1 - \cos^2\theta}{\cos^2\theta} \, d\theta = \int_0^{\pi/2} \sec^2\theta - 1 \, d\theta$$

$$= \tan\theta - \theta \Big|_0^{\pi/2} = \infty$$

(m) $\displaystyle\int_0^1 r\log r\,dr = \frac{r^2}{2}\log r\Big|_0^1 - \int_0^1 \frac{r}{2}\,dr$

$$= \left(\frac{r^2}{2}\log r - \frac{r^2}{4}\right)\Bigg|_0^1$$

Temos, por L'Hôpital,

$$\lim_{r\to 0+} r^2\log r = \lim_{r\to 0+} \frac{\log r}{1/r^2}$$

$$= \lim_{r\to 0+} \frac{1/r}{-2/r^3} = \lim_{r\to 0+} -\frac{1}{2}r^2 = 0$$

$$\Rightarrow \int_0^1 r\log r\,dr = -\frac{1}{4}$$

(n) Temos que

$$\int \frac{e^{1/x}}{x^3}\,dx \quad \left\{\begin{array}{l} u = x^{-1} \\ \Rightarrow du = x^{-2}dx \\ \Rightarrow dx = u^{-2}du \end{array}\right.$$

$$= \int_{-1} \frac{u^3e^u}{u^2}\,du = \int ue^u\,du = ue^u - \int e^u\,du$$

$$= (u-1)e^u = \left(\frac{1}{x}-1\right)e^{1/x} = \frac{1-x}{x}e^{1/x}$$

Logo,

$$\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx = \lim_{\varepsilon \to 0} \left. \frac{1-x}{x}\, e^{1/x} \right|_{-1}^{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} (1-\varepsilon)\, \frac{e^{1/\varepsilon}}{\varepsilon} + 2e^{-1} = \infty$$

(o) $$\int_{0}^{\pi/2} \frac{\cos\theta}{\sqrt{\operatorname{sen}\theta}}\, d\theta = 2\sqrt{\operatorname{sen}\theta} \,\Big|_{0}^{\pi/2} = 2$$

Exercício 4. Determine se as seguintes integrais convergem ou divergem:

(a) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ (b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$ (c) $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

(d) $\displaystyle\int_1^\infty \frac{1+\operatorname{sen}^2 x}{\sqrt{x}}\,dx$ (e) $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ (f) $\displaystyle\int_0^\pi \frac{\operatorname{sen}^2 x}{\sqrt{x}}\,dx$

(a) Converge pois

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_0^\infty \frac{x}{x^3+1}\,dx$$

$$\leq \int_0^1 dx + \int_1^\infty \frac{x}{x^3}\,dx$$

$$= 1 - \frac{1}{x}\Big|_1^\infty = 2$$

(b) Como $x \geq 1$, temos

$$\frac{x+1}{\sqrt{x^4-x}} = \frac{x+1}{\sqrt{x}\sqrt{x^2+x+1}} \qquad \frac{1}{\sqrt{x-1}}$$

$$\geq \frac{x}{\sqrt{x}\cdot\sqrt{3x^2}\cdot\sqrt{x}} = \frac{x}{\sqrt{3}\cdot x^2}$$

$$= \frac{1}{\sqrt{3}}\cdot\frac{1}{x}$$

$$\Rightarrow \int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx \geq \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$$

$$\geq \int_2^\infty \frac{1}{\sqrt{3}}\cdot\frac{dx}{x} = \infty$$

(c) $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx \geq \int_0^1 \frac{dx}{x^{3/2}} = -2x^{-1/2}\Big|_0^1 = \infty$

(d) $\displaystyle\int_1^\infty \frac{1+\operatorname{sen}^2 x}{\sqrt{x}}\,dx \geq \int_1^\infty \frac{dx}{x^{1/2}} = 2x^{1/2}\Big|_1^\infty = \infty$

(e) $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx \leq \int_0^\infty \frac{\pi/2}{e^x}\,dx$

$\displaystyle = -\frac{\pi}{2}e^{-x}\Big|_0^\infty = \frac{\pi}{2}$

Portanto, a integral converge.

(f) $\displaystyle\int_0^\pi \frac{\operatorname{sen}^2 x}{\sqrt{x}}\,dx = \int_0^1 \frac{\operatorname{sen}^2 x}{\sqrt{x}}\,dx + \int_1^\pi \frac{\operatorname{sen}^2 x}{\sqrt{x}}\,dx$

O problema está apenas na 1ª integral.

$\displaystyle\int_0^1 \frac{\operatorname{sen}^2 x}{\sqrt{x}}\,dx \leq \int_0^1 \frac{\operatorname{sen}^2 x}{x}\,dx$

O integrando é contínuo em $x=0$:

$\displaystyle\lim_{x\to 0} \frac{\operatorname{sen}^2 x}{x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{x}\cdot \operatorname{sen} x = 0$

Assim, a integral converge.

Exercício 5. Calcule:

(a) $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx$ (b) $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx$

(a) $$\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^1 \frac{dx}{\sqrt{x}(1+x)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)}$$

$u = \sqrt{x} \Rightarrow du = \frac{1}{2u}dx \Rightarrow dx = 2u\,du$

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{2\cancel{u}\,du}{\cancel{u}(1+u^2)} = 2\arctan u$$

$$= 2\arctan\sqrt{x}$$

Então

$$\int_0^1 \frac{dx}{\sqrt{x}(1+x)} = 2\arctan\sqrt{x}\Big|_0^1 = \frac{\pi}{2}$$

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(1+x)} = 2\arctan\sqrt{x}\Big|_1^{\infty} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Logo,

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)} = \pi$$

(b) $$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}}$$

$x = 2u \Rightarrow dx = 2u\,du$

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{\cancel{2}du}{\cancel{2}u\sqrt{4u^2-4}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}}$$

$u = \sec w \Rightarrow du = \sec w \cdot \tan w \, dw$

$$= \frac{1}{2}\int \frac{\sec w \cdot \tan w}{\sec w \cdot \tan w}\, dw = \frac{w}{2}$$

$\sec w = u = \frac{x}{2}$

$\Rightarrow \tan^2 w = \sec^2 w - 1 = \frac{x^2-4}{4}$

$\Rightarrow \tan w = \frac{\sqrt{x^2-4}}{2} \Rightarrow w = \arctan \frac{\sqrt{x^2-4}}{2}$

Logo,

$$\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\arctan \frac{\sqrt{x^2-4}}{2}$$

Assim,

$$\int_2^3 \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\arctan \frac{\sqrt{x^2-4}}{2}\Bigg|_2^3$$

$$= \frac{1}{2}\arctan \frac{\sqrt{5}}{2}$$

$$\int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\arctan\frac{\sqrt{x^2-4}}{2}\Bigg|_3^\infty$$

$$= \frac{\pi}{2} - \frac{1}{2}\arctan\frac{\sqrt{5}}{2}$$

Portanto,

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \frac{\pi}{2}$$

Exercício 6. Encontre os valores de p para os quais a integral converge e calcule o valor da integral para esses valores de p.

(a) $\int_0^1 \frac{dx}{x^p}$ (b) $\int_0^1 x^p \log x\, dx$ (c) $\int_e^\infty \frac{dx}{x(\log x)^p}$

(a) $$\int_0^1 x^{-p}dx = \begin{cases} \frac{x^{1-p}}{1-p}\Big|_0^1 & \text{se } p \neq 1 \\ \log\Big|_0^1 & \text{se } p = 1 \end{cases},$$

que converge somente se $p<1$.
Nesse caso, temos

$$\int_0^1 x^{-p}dx = \frac{1}{1-p}$$

(b) Se $p \neq -1$,

$$\int_0^1 x^p \log x\, dx = \frac{x^{p+1}}{p+1}\log x\Big|_0^1 - \int_0^1 \frac{x^p}{p+1}dx$$

$$= \frac{x^{p+1}\log x}{p+1}\Big|_0^1 - \frac{x^{p+1}}{(p+1)^2}\Big|_0^1$$

$$= \lim_{x\to 0+} \frac{-x^{p+1}\log x}{p+1} - \frac{1}{(p+1)^2}$$

Temos, por L'Hôpital,

$$\lim_{x\to 0+} -\frac{\log x}{x^{-(p+1)}} = \lim_{x\to 0+} \frac{x^{-1}}{(p+1)x^{-(p+2)}}$$

$$= \lim_{x \to 0+} \frac{x^{p+1}}{p+1} = 0$$

Assim,

$$\int_0^1 x^p \log x \, dx = \frac{-1}{(p+1)^2} \quad \text{se } p \neq -1.$$

Quando $p = -1$, temos

$$\int \frac{\log x}{x} \, dx = \frac{1}{2} (\log x)^2$$

Assim,

$$\int_0^1 x^{-1} \log x \, dx = - \lim_{x \to 0+} (\log x)^2 = -\infty \quad (p = -1)$$

Exercício 7. Calcule, para todo $n \in \mathbb{N}$, $\int_0^\infty x^n e^{-x}\, dx$

$$\int_0^\infty x^n e^{-x} dx = -x^n e^{-x}\Big|_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx$$

$$= n\int_0^\infty x^{n-1} e^{-x} dx$$

A iteração desse argumento leva a

$$\int_0^\infty x^n e^{-x} dx = n!\int_0^\infty e^{-x} dx = n!\left(-e^{-x}\right)\Big|_0^\infty$$

$$= n!$$

Exercício 8.

(a) Mostre que $\int_{-\infty}^{\infty} x\,dx$ é divergente.

(b) Verifique que $\lim\limits_{t\to\infty} \int_{-t}^{t} x\,dx = 0$.

Isso mostra que **NÃO** podemos definir $\int_{-\infty}^{\infty} x\,dx = \lim\limits_{t\to\infty} \int_{-t}^{t} x\,dx$.

(a) $$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$$

Temos

$$\int_{-\infty}^{0} x\,dx = \frac{x^2}{2}\Big|_{-\infty}^{0} = -\infty$$

e

$$\int_{0}^{\infty} x\,dx = \frac{x^2}{2}\Big|_{0}^{\infty} = \infty,$$

logo $\int_{-\infty}^{\infty} x\,dx$ diverge.

(b) $\lim\limits_{t\to\infty} \int_{-t}^{t} x\,dx = 0$ pois $f(x) = x$

é função ímpar.

Exercício 9. Se $f(t)$ é contínua para $t \geqslant 0$, sua transformada de Laplace é a função F definida por

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

e o seu domínio é o conjunto dos números s para os quais a integral converge. Calcule as transformadas de Laplace das seguintes funções:

(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

(a) $F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \quad (s > 0)$

(b) $F(s) = \int_0^\infty e^{-(s-1)t}dt = \frac{1}{s-1} \quad (s > 1)$

(c) $F(s) = \int_0^\infty te^{-st}dt = -\frac{1}{s}te^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt$

$= -\frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2} \quad (s > 0)$

Exercício 10. Mostre que se $|f(t)| \leqslant Me^{at}$, então a transformada de Laplace $F(s)$ de f existe para $s > a$.

$$|F(s)| \leq \int_0^\infty |f(t)|\, e^{-st}\, dt \leq M \int_0^\infty e^{-(s-a)t}\, dt$$

$$= M\left[-\frac{1}{(s-a)} e^{-(s-a)t}\right]\Bigg|_0^\infty$$

$$= \frac{M}{s-a} < \infty \quad \text{se } s > a.$$

Exercício 11. Mostre que

$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

$$\int_0^\infty x^2 e^{-x^2} dx = -\frac{1}{2} x e^{-x^2}\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2} dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x^2} dx$$

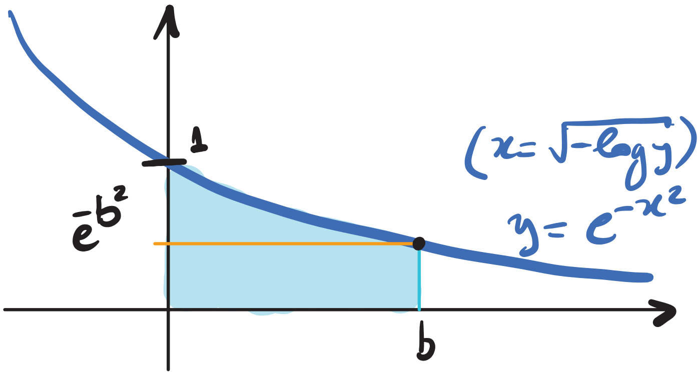
Exercício 12. Mostre que

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 \sqrt{-\log y}\,dy$$

Note que $y = e^{-x^2} \Rightarrow -x^2 = \log y \Rightarrow x = \sqrt{-\log y}$

Além disso, $\forall\, b > 0$,

$$\int_0^b e^{-x^2}dx = \int_{e^{-b^2}}^1 \sqrt{-\log y}\,dy + b \cdot e^{-b^2}$$



Logo,

$$\int_0^\infty e^{-x^2}dx = \lim_{b\to\infty}\int_0^b e^{-x^2}dx = \lim_{b\to\infty}\int_{e^{-b^2}}^1 \sqrt{-\log y}\,dy + \lim_{b\to\infty}\frac{b}{e^{b^2}} = \int_0^1 \sqrt{-\log y}\,dy.$$

Exercício 13. Encontre o valor da constante C para o qual a integral converge e calcule o valor da integral:

(a) $\displaystyle\int_0^{\infty}\left(\frac{1}{\sqrt{x^2+4}}-\frac{C}{x+2}\right)dx$ (b) $\displaystyle\int_0^{\infty}\left(\frac{x}{x^2+1}-\frac{C}{3x+1}\right)dx$

(a) Temos

$x = 2u \Rightarrow dx = 2\,du$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{du}{\sqrt{u^2+1}} = \int \frac{\sec^{\cancel{2}} v\, dv}{\cancel{\sec v}}$$

$u = \tan v \Rightarrow du = \sec^2 v\, dv$

$$= \int \frac{\sec v(\tan v + \sec v)}{\sec v + \tan v}dv = \int \frac{\sec v \tan v + \sec^2 v}{\sec v + \tan v}dv$$

$$= \log(\sec v + \tan v) = \log\left(u + \sqrt{u^2+1}\right)$$

$$= \log\left(\frac{x}{2} + \frac{\sqrt{x^2+4}}{2}\right)$$

Por outro lado,

$$\int \frac{dx}{x+2} = \log(x+2)$$

Assim,

$$\int_0^b \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx = \log\left(\frac{x}{2} + \frac{\sqrt{x^2+4}}{2}\right)\Bigg|_0^b$$

$$- C\log(x+2)\Big|_0^b =$$

$$= \log\left(b+\sqrt{b^2+4}\right) - \log 2 - \cancel{\log 2} + \cancel{\log 2}$$
$$- c\log(b+2) + c\log 2$$
$$= \log\left[\frac{b+\sqrt{b^2+4}}{(b+2)^c}\right] + (c-1)\log 2$$

Precisamos ter

$$0 < \lim_{b\to\infty} \frac{b+\sqrt{b^2+4}}{(b+2)^c} < \infty$$

Note que é preciso ter $c>0$. Além disso,

$$\frac{b+\sqrt{b^2+4}}{(b+2)^c} = \frac{b\cdot\left(1+\sqrt{1+4/b^2}\right)}{(b+2)^c}$$
$$= \frac{1+\sqrt{1+4/b^2}}{b^{-1}(b+2)^c} = \frac{1+\sqrt{1+4/b^2}}{\left[b^{-1/c}(b+2)\right]^c}$$
$$= \frac{1+\sqrt{1+4/b^2}}{\left[b^{1-1/c}+2/b^{1/c}\right]^c}$$

O numerador tende a 2 se $b\to\infty$.
Já o denominador tem o limite

$$\lim_{b\to\infty}\left(b^{1-1/c}+2/b^{1/c}\right)^c = \left(\lim_{b\to\infty} b^{1-\frac{1}{c}}\right)^c,$$

que só pode ser 0, 1 ou ∞,

dependendo se $0<C<1$, $C=1$ ou $C>1$, respectivamente. Portanto, o único valor que nos serve é $C=1$, e aí

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{1}{x+2} \right) dx = \lim_{b\to\infty} \log \left[\frac{b+\sqrt{b^2+4}}{(b+2)^C} \right]$$

$$= \lim_{b\to\infty} \log \left(\frac{1+\sqrt{1+4/b^2}}{1+2/b} \right) = \log 2$$

Exercício 14. Suponha que f é contínua em $[0,\infty)$ e que $\lim_{x\to\infty} f(x) = 1$. É possível que $\int_0^\infty f(x)\,dx$ seja convergente?

Não. Como $\lim_{x\to\infty} f(x) = 1$, existe $a>0$ tal que se $x>a$ então

$$\frac{1}{2} < f(x) < \frac{3}{2}$$

Daí,

$$\int_a^\infty f(x)\,dx \geqslant \int_a^\infty \frac{1}{2}\,dx = \infty$$

Isso implica que $\int_0^\infty f(x)\,dx$ diverge.

Exercício 15. Prove que $\int_0^\infty \sin^2\left[\pi\left(x+\frac{1}{x}\right)\right] dx$ não existe.

A ideia é mostrar que

$$\operatorname{sen} \pi\left(x+\frac{1}{x}\right) > \operatorname{sen}\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

em intervalos regulares. Para todo $k \in \mathbb{N}$, isso ocorre se

$$2k\pi + \frac{\pi}{4} < \pi\left(x+\frac{1}{x}\right) < 2k\pi + \frac{3\pi}{4}$$

Note que $\alpha < x + \frac{1}{x}$ se e só se

$x^2 - \alpha x + 1 > 0$, o que ocorre se

$$x > \frac{\alpha + \sqrt{\alpha^2 - 4}}{2}$$

Por outro lado, $x + \frac{1}{x} < \beta$ se e só se

$x^2 - \beta x + 1 < 0$, o que ocorre se

$$x < \frac{\beta + \sqrt{\beta^2 - 4}}{2}$$

Para $\beta = \alpha + \frac{1}{2}$, a condição é

$$x < \frac{\alpha + \frac{1}{2} + \sqrt{\alpha^2 + \alpha + \frac{1}{4} - 4}}{2},$$

que se cumpre se

$$x < \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} + \frac{1}{4}$$

Assim, para $k \in \mathbb{N}$, $k \geq 2$, defina

$$\alpha_k = 2k + \frac{1}{4}$$

Então, pelo que vimos antes, se

$$\frac{\alpha_k + \sqrt{\alpha_k^2 - 4}}{2} < x < \frac{\alpha_k + \sqrt{\alpha_k^2 - 4}}{2} + \frac{1}{4}$$

então

$$\pi\alpha_k < \pi\left(x + \frac{1}{x}\right) < \pi\alpha_k + \frac{\pi}{2},$$

ou seja,

$$2k\pi + \frac{\pi}{4} < \pi\left(x + \frac{1}{x}\right) < 2k\pi + \frac{3\pi}{4}$$

$$\Rightarrow \operatorname{sen} \pi\left(x + \frac{1}{x}\right) > \frac{1}{\sqrt{2}}.$$

Assim, se $\beta_k = \dfrac{\alpha_k + \sqrt{\alpha_k^2 - 4}}{2}$,

$$\int_0^\infty \operatorname{sen}^2\left[\pi\left(x + \frac{1}{x}\right)\right] dx \geq \sum_{k=1}^{\infty} \int_{\beta_k}^{\beta_k + 1/4} \frac{1}{2}\, dx$$

$$= \sum_{k=1}^{\infty} \frac{1}{8} = \infty.$$

Exercício 16. Calcule $\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6} \right)^2 dx$

$$u = x^3 \Rightarrow du = 3x^2 dx$$
$$\Rightarrow dx = \frac{1}{3} u^{-2/3} du$$

$$\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6} \right)^2 dx = \int_{-1}^{\infty} \frac{u^{8/3} \cdot u^{-2/3}}{(1+u^2)^2} du$$

$$= \int_{-1}^{\infty} \frac{u^2}{(1+u^2)^2} du$$

Usando frações parciais, temos

$$\frac{u^2}{(1+u^2)^2} = \frac{a}{1+u^2} + \frac{bu+c}{(1+u^2)^2}$$
$$= \frac{au^2 + bu + a + c}{(1+u^2)^2}$$

$$\Rightarrow \begin{cases} a = 1 \\ b = 0 \\ a + c = 0 \Rightarrow c = -a = -1 \end{cases}$$

Logo,

$$\int_{-1}^{\infty} \frac{u^2}{(1+u^2)^2} du = \int_{-1}^{\infty} \frac{1}{1+u^2} - \frac{1}{(1+u^2)^2} du$$

$$= \arctan u \Big|_{-1}^{\infty} - \int_{-1}^{\infty} \frac{du}{(1+u^2)^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{4} - \int_{-1}^{\infty} \frac{du}{(1+u^2)^2} = \frac{3\pi}{4} - \int_{-1}^{\infty} \frac{du}{(1+u^2)^2}$$

Vamos calcular a integral:

$$u = \tan\theta \Rightarrow du = \sec^2\theta \, d\theta$$

$$\int_{-1}^{\infty} \frac{du}{(1+u^2)^2} = \int_{-\pi/4}^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^4\theta \, d\theta} = \int_{-\pi/4}^{\pi/2} \cos^2\theta \, d\theta$$

$$2\cos^2\theta - 1 = \cos^2\theta - \operatorname{sen}^2\theta = \cos 2\theta$$

$$\Rightarrow \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$= \int_{-\pi/4}^{\pi/2} \frac{\cos 2\theta - 1}{2} \, d\theta = \frac{\operatorname{sen} 2\theta}{4} - \frac{\theta}{2} \Big|_{-\pi/4}^{\pi/2}$$

$$= \frac{1}{4} - \frac{\pi}{4} - \frac{\pi}{8} = \frac{1}{4} - \frac{3\pi}{8}$$

Logo,

$$\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6}\right)^2 dx = \frac{3\pi}{4} - \frac{1}{4} + \frac{3\pi}{8}$$

$$= \frac{9\pi}{8} - \frac{1}{4}$$

Exercício 17. Prove que $\lim_{k\to\infty} \int_0^\infty \frac{dx}{1+kx^{10}} = 0$

Seja $\varepsilon > 0$. Temos

$$\int_0^\infty \frac{dx}{1+kx^{10}} = \int_0^\varepsilon \frac{dx}{1+kx^{10}} + \int_\varepsilon^\infty \frac{dx}{1+kx^{10}}$$

Se $\varepsilon < x < \infty$,

$$\int_\varepsilon^\infty \frac{dx}{1+kx^{10}} \leq \int_\varepsilon^\infty \frac{dx}{kx^{10}} = -\frac{1}{9k} \cdot x^{-9} \Big|_\varepsilon^\infty$$

$$= \frac{1}{9k} \cdot \frac{1}{\varepsilon^9}$$

Se $0 < x < \varepsilon$,

$$\int_0^\varepsilon \frac{dx}{1+kx^{10}} \leq \int_0^\varepsilon dx = \varepsilon$$

Logo,

$$0 \leq \int_0^\infty \frac{dx}{1+kx^{10}} \leq \varepsilon + \frac{1}{9k\varepsilon^9} \xrightarrow[k\to\infty]{} \varepsilon$$

Ou seja, $\forall\ \varepsilon > 0$,

$$0 \leq \int_0^\infty \frac{dx}{1+kx^{10}} \leq \varepsilon$$

Como $\varepsilon > 0$ é arbitrário, fazendo $\varepsilon \to 0$ segue que

$$\int_0^\infty \frac{dx}{1+kx^{10}} = 0$$

Exercício 18. Mostre que se $a > 0$, então

$$\lim_{h\to 0_+} \int_{-a}^{a} \frac{h}{h^2+x^2}\,dx = \pi$$

Temos

$x = hu \Rightarrow dx = h\,du$

$$\int_{-a}^{a} \frac{h}{h^2+x^2}\,dx = \int_{-a/h}^{a/h} \frac{\cancel{h^2}\,du}{\cancel{h^2}(1+u^2)}$$

$$= \arctan u \Big|_{-a/h}^{a/h} = 2\cdot \arctan \frac{a}{h}$$

$$\xrightarrow[h\to 0_+]{} 2\cdot\frac{\pi}{2} = \pi$$

Exercício 19. Se $\int_a^\infty \frac{f(x)}{x}dx$ converge para todo $a > 0$ e se $\lim\limits_{x\to 0} f(x) = L$, mostre que

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx$$

converge para α e β positivos e tem o valor $L \log \frac{\beta}{\alpha}$.

Temos

$u = \alpha x \Rightarrow du = \alpha\, dx$

$$\int_a^\infty \frac{f(\alpha x)}{x}\,dx = \int_{\alpha a}^\infty \frac{f(u)\,\cancel{\alpha^{-1}}du}{u\cdot\cancel{\alpha^{-1}}} = \int_{\alpha a}^\infty \frac{f(u)}{u}\,du$$

Analogamente,

$$\int_a^\infty \frac{f(\beta x)}{x}\,dx = \int_{\beta a}^\infty \frac{f(u)}{u}\,du$$

Temos que

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx = \lim_{a\to 0^+} \int_a^\infty \frac{f(\alpha x) - f(\beta x)}{x}\,dx$$

$$= \lim_{a\to 0^+}\left[\int_{\alpha a}^\infty \frac{f(u)}{u}\,du - \int_{\beta a}^\infty \frac{f(u)}{u}\,du\right]$$

Se $\alpha \leq \beta$, temos $a\alpha \leq a\beta$ para todo $a > 0$, e assim

$$\int_{\alpha a}^\infty \frac{f(u)}{u}\,du - \int_{\beta a}^\infty \frac{f(u)}{u}\,du = \int_{\alpha a}^{\beta a} \frac{f(u)}{u}\,du$$

Se $\beta < \alpha$, obtemos

$$\int_{\alpha a}^{\infty} \frac{f(u)}{u}\,du - \int_{\beta a}^{\infty} \frac{f(u)}{u}\,du = -\int_{\beta a}^{\alpha a} \frac{f(u)}{u}\,du$$

$$= \int_{\alpha a}^{\beta a} \frac{f(u)}{u}\,du$$

Portanto, em qualquer caso,

$$\int_0^{\infty} \frac{f(\alpha x) - f(\beta x)}{x}\,dx = \lim_{a \to 0+} \int_{\alpha a}^{\beta a} \frac{f(u)}{u}\,du$$

Sem perda de generalidade vamos assumir que $\alpha \leq \beta$.

Seja $\varepsilon > 0$. Como $L = \lim_{x \to 0} f(x)$, existe $\delta > 0$ tal que

$$L - \varepsilon < f(x) < L + \varepsilon \quad \text{se } 0 < x < \delta$$

Tome $a > 0$ tal que

$$a < \delta / \beta$$

Então

$$\int_{a\alpha}^{a\beta} \frac{L - \varepsilon}{u}\,du < \int_{a\alpha}^{a\beta} \frac{f(u)}{u}\,du < \int_{a\alpha}^{a\beta} \frac{L + \varepsilon}{u}\,du$$

$$\Rightarrow (L-\varepsilon) \log \frac{a\beta}{a\alpha} < \int_{a\alpha}^{a\beta} \frac{f(u)}{u}\, du$$

$$< (L+\varepsilon) \log \frac{a\beta}{a\alpha}$$

Assim, $\forall\, \varepsilon > 0$,

$$(L-\varepsilon) \log \frac{\beta}{\alpha} \leq \lim_{a \to 0^+} \int_{a\alpha}^{a\beta} \frac{f(u)}{u}\, du$$

$$\leq (L+\varepsilon) \log \frac{\beta}{\alpha}$$

Logo, como $\varepsilon > 0$ é arbitrário, segue que

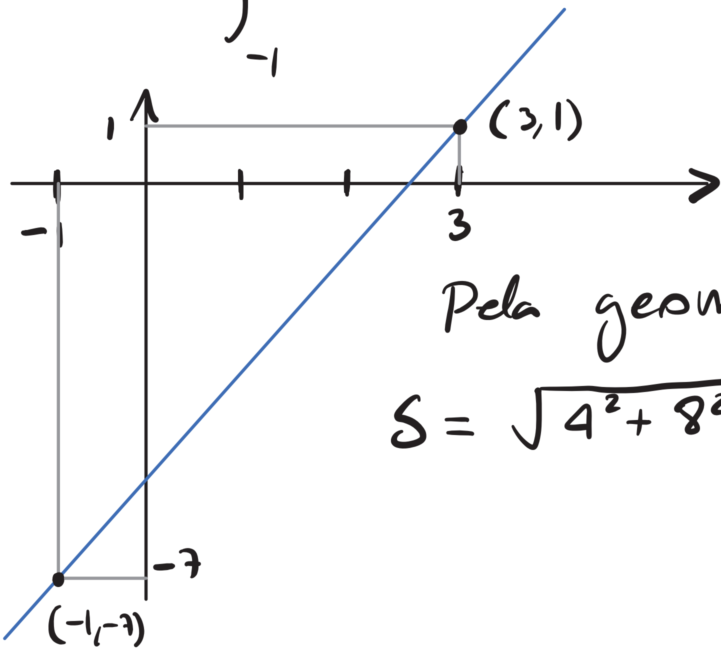
$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x}\, dx = \lim_{a \to 0^+} \int_{a\alpha}^{a\beta} \frac{f(u)}{u}\, du$$

$$= L \cdot \log \frac{\beta}{\alpha}.$$

Exercício 1. Usando a fórmula de comprimento de arco, calcule o comprimento de arco das seguintes curvas e compare com o resultado da geometria básica:

(a) $y = 2x - 5, \quad x \in [-1, 3]$ (b) $y = \sqrt{2 - x^2}, \quad x \in [0, 1]$

(a) $S = \int_{-1}^{3} \sqrt{1 + 2^2}\, dx = 4\sqrt{5}$



Pela geometria,

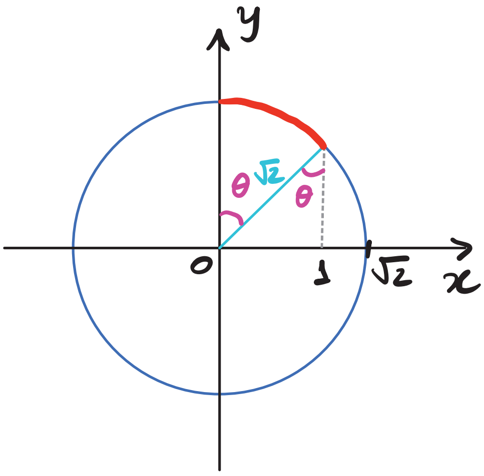
$S = \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$

(b) $S = \int_0^1 \sqrt{\frac{x^2}{2 - x^2} + 1}\, dx$

$= \int_0^1 \frac{\sqrt{2}}{\sqrt{2 - x^2}}\, dx$ $\left\{ \begin{array}{l} x = \sqrt{2}u \\ dx = \sqrt{2}\, du \end{array} \right.$

$= \sqrt{2} \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - u^2}}\, du = \sqrt{2} \operatorname{arc\,sen} \frac{1}{\sqrt{2}} = \frac{\pi\sqrt{2}}{4}$

Vamos analisar pela geometria:



$$y = \sqrt{2-x^2}$$
$$\Rightarrow y^2 = 2 - x^2$$
$$\Rightarrow x^2 + y^2 = (\sqrt{2})^2$$

Geometricamente calculamos o arco subentendido pelo ângulo θ no círculo de raio $\sqrt{2}$, onde

$$\operatorname{sen}\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

Daí, $S = \theta r = \dfrac{\pi\sqrt{2}}{4}$.

Exercício 2. Encontre o comprimento das seguintes curvas:

(a) $y = 1 + 6x^{3/2}$, $x \in [0, 1]$

(b) $36y^2 = (x^2 - 4)^3$, $x \in [2, 3]$, $y \geqslant 0$

(c) $y = \log(\cos x)$, $x \in [0, \frac{\pi}{3}]$

(d) $y = \log(\sec x)$, $x \in [0, \frac{\pi}{4}]$

(e) $y = 3 + \frac{1}{2}\cosh 2x$, $x \in [0, 1]$

(f) $y = \sqrt{x - x^2} + \operatorname{arcsen}\sqrt{x}$

(g) $y = \log(1 - x^2)$, $x \in [0, \frac{1}{2}]$

(a) $$S = \int_0^1 \sqrt{1 + 81x}\, dx = \frac{2}{243}(1 + 81\cdot x)^{3/2}\Big|_0^1$$

$$= \frac{2}{243}\left[(82)^{3/2} - 1\right]$$

(b) $$y = \frac{1}{6}(x^2 - 4)^{3/2}$$

$$y' = \frac{1}{2}x(x^2 - 4)^{1/2}$$

$$S = \int_2^3 \sqrt{\frac{x^2(x^2-4) + 4}{2}}\, dx$$

$$= \int_2^3 \sqrt{\frac{x^4 - 4x^2 + 4}{2}}\, dx$$

$$= \int_2^3 \frac{x^2 - 2}{2}\, dx = \frac{x^3}{6} - x\Big|_2^3$$

$$= \frac{9}{2} - 3 - \frac{4}{3} + 2 = \frac{19}{6} - \frac{6}{6} = \frac{13}{6}$$

(c) $y = \log(\cos x)$

$$\Rightarrow y' = -\frac{\operatorname{sen} x}{\cos x} = -\tan x$$

$$S = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/3} \sec x \, dx$$

$$= \int_0^{\pi/3} \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

$$= \int_0^{\pi/3} \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x}\, dx$$

$$= \log(\sec x + \tan x)\Big|_0^{\pi/3}$$

$$= \log(2+\sqrt{3})$$

(d) $y = \log(\sec x) = \log\left(\frac{1}{\cos x}\right)$

$$= -\log(\cos x)$$

Logo, são as mesmas contas do item (c):

$$S = \log(\sec x + \tan x)\Big|_0^{\pi/4} = \log\left(\frac{1}{\sqrt{2}} + 1\right)$$

(e) $y = 3 + \frac{1}{2}\cosh 2x$

$\Rightarrow y' = \operatorname{senh} 2x$

$$S = \int_0^1 \sqrt{\operatorname{senh}^2 2x + 1}\, dx$$

$$= \int_0^1 \cosh 2x\, dx = \left.\frac{\operatorname{senh} 2x}{2}\right|_0^1$$

$$= \frac{\operatorname{senh} 2}{2} = \frac{e^2 - e^{-2}}{4}$$

(f) $y = \sqrt{x - x^2} + \operatorname{arsen}\sqrt{x}$

Só está definida em $[0, 1]$.

$$y' = \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}} \frac{1-x}{\sqrt{1-x}} = \sqrt{\frac{1-x}{x}}$$

$$S = \int_0^1 \sqrt{\frac{1-\cancel{x}+\cancel{x}}{x}}\, dx = \int_0^1 \frac{dx}{\sqrt{x}}$$

$$= 2\sqrt{x}\Big|_0^1 = 2$$

(g) $y = \log(1-x^2)$

$$y' = \frac{-2x}{1-x^2}$$

$$S = \int_0^{1/2} \sqrt{\frac{4x^2 + (1-x^2)^2}{1-x^2}}\, dx$$

$$= \int_0^{1/2} \sqrt{\frac{x^4 + 2x^2 + 1}{1-x^2}}\, dx$$

$$= \int_0^{1/2} \frac{1+x^2}{1-x^2}\, dx = \int_0^{1/2} \frac{x^2 - 1 + 2}{1-x^2}\, dx$$

$$= \int_0^{1/2} -1 + \frac{2}{(1-x)(1+x)}\, dx$$

$$= \int_0^{1/2} -1 + \frac{1}{1-x} + \frac{1}{1+x}\, dx$$

$$= -x - \log(1-x) + \log(1+x) \Big|_0^{1/2}$$

$$= \log\frac{3}{2} - \log\frac{1}{2} - \frac{1}{2}$$

$$= \log 3 - \log 2 + \log 2 - \frac{1}{2}$$

$$= \log 3 - \frac{1}{2}$$

Exercício 3. Para a função $f(x) = \frac{1}{4}e^{x} + e^{-x}$, demonstre que o comprimento de arco em qualquer intervalo tem o mesmo valor que a área sob a curva.

$$f'(x) = \frac{1}{4}e^{x} - e^{-x}$$

$$S = \int_a^b \sqrt{\frac{e^{2x}}{16} - \frac{1}{2} + e^{-2x} + 1}\, dx$$

$$= \int_a^b \sqrt{\frac{e^{2x}}{16} + \frac{1}{2} + e^{-2x}}\, dx = \int_a^b \left(\frac{e^{x}}{4} + e^{-x}\right) dx$$

$$= \int_a^b f(x)\, dx$$

Logo, o comprimento de arco corresponde à área.

Exercício 4. Encontre a função de comprimento de arco para a curva $y = \operatorname{arcsen} x + \sqrt{1 - x^2}$ com ponto inicial $(0, 1)$.

$$y' = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sqrt{\frac{1-x}{1+x}}$$

$$S(x) = \int_0^x \sqrt{\frac{1-t}{1+t} + 1}\, dt = \int_0^x \frac{\sqrt{2}}{\sqrt{1+t}}\, dt$$

$$= 2\sqrt{2}\,\sqrt{1+t}\,\Big|_0^x = 2\sqrt{2}\left(\sqrt{1+x} - 1\right)$$

Exercício 5. Um vento contínuo sopra uma pipa para oeste. A altura da pipa acima do solo a partir da posição horizontal $x = 0$ até $x = 25$ é dada por $y = 50 - 0,1(x - 15)^2$. Ache a distância percorrida pela pipa.

Temos que calcular o comprimento de arco entre $x = 0$ e $x = 25$ da curva $y = 50 - \frac{1}{10}(x-15)^2$.

$$y' = \frac{1}{5}(x-15)$$

$$S = \int_0^{25} \sqrt{\left(\frac{x-15}{5}\right)^2 + 1}\, dx$$

$$\operatorname{senh} u = \frac{x-15}{5} \Rightarrow \cosh u \, du = \frac{dx}{5}$$

$$S = \int_{\operatorname{arsenh}(-3)}^{\operatorname{arsenh}(1/2)} \cosh^2 u \, du = \int_{\operatorname{arsenh}(-3)}^{\operatorname{arsenh}(1/2)} \frac{\cosh 2u + 1}{2}\, du$$

$$= \frac{\operatorname{senh} 2u}{4} + u \Big|_{\operatorname{arsenh}(-3)}^{\operatorname{arsenh}(1/2)}$$

$$= \frac{\operatorname{senh} u \sqrt{1 + \operatorname{senh}^2 u}}{2} + u \Big|_{\operatorname{arsenh}(-3)}^{\operatorname{arsenh}(1/2)}$$

$$= \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{5}}{2} + \frac{3}{2} \cdot \sqrt{2}\sqrt{5}\right) + \log\left(x + \sqrt{x^2+1}\right)\Big|_{-3}^{1/2}$$

$$= \frac{\sqrt{5}}{8}\left(1 + 12\sqrt{2}\right) + \log\left(\frac{1 + \sqrt{5}}{2}\right)$$

$$- \log\left(-3 + \sqrt{10}\right)$$

Exercício 6. Um fio pendurado entre os postes localizados nas retas $x = -b$ e $x = b$ adota o formato de uma catenária, dada pela equação $y = c + a\cosh(x/a)$. Calcule o comprimento do fio.

$$y = c + a\cosh\left(\frac{x}{a}\right)$$

$$\Rightarrow y' = \operatorname{senh}\left(\frac{x}{a}\right)$$

$$S = \int_{-b}^{b} \sqrt{\operatorname{senh}^2\left(\frac{x}{a}\right) + 1}\, dx$$

$$= \int_{-b}^{b} \cosh \frac{x}{a}\, dx = a \operatorname{senh} \frac{x}{a} \Big|_{b}^{b}$$

$$= 2a \operatorname{senh} \frac{b}{a}$$

Exercício 7. Prove que a curva definida por

$$y = \begin{cases} x^2 \operatorname{sen} \dfrac{1}{x} & \text{se } 0 < x \leqslant 1 \\ 0 & \text{se } x = 0 \end{cases}$$

tem comprimento finito, ao passo que a curva contínua definida por

$$y = \begin{cases} x \operatorname{sen} \dfrac{1}{x} & \text{se } 0 < x \leqslant 1 \\ 0 & \text{se } x = 0 \end{cases}$$

não é retificável (ou seja, tem comprimento infinito).

Vejamos a primeira curva.

$$y = x^2 \operatorname{sen} \frac{1}{x}$$

$$\Rightarrow y' = 2x \operatorname{sen} \frac{1}{x} - \frac{x^2}{x^2} \cos \frac{1}{x}$$

$$= 2x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$$

Assim,

$$S = \int_0^1 \sqrt{4x^2 \operatorname{sen}^2 \frac{1}{x} + \cos^2 \frac{1}{x} - 4x \operatorname{sen} \frac{1}{x} \cos \frac{1}{x} + 1}\, dx$$

$$= \int_0^1 \sqrt{4x^2 \operatorname{sen}^2 \frac{1}{x} + \cos^2 \frac{1}{x} - 2x \operatorname{sen} \frac{2}{x} + 1}\, dx$$

$$= \int_0^1 \sqrt{(4x^2 - 1) \operatorname{sen}^2 \frac{1}{x} - 2x \operatorname{sen} \frac{2}{x} + 2}\, dx$$

$$\leq \int_0^1 \sqrt{4x^2 - 2x + 2}\, dx$$

$$= \int_0^1 \sqrt{(2x-1)^2 + 1}\, dx \qquad (2x-1)^2 \leq 1 \text{ em } [0,1]$$

$$\leq \int_0^1 \sqrt{2}\, dx = \sqrt{2}$$

Logo, S é finito.

Vejamos agora a 2ª curva.

$$y = x \operatorname{sen} \frac{1}{x}$$

$$\Rightarrow y' = \operatorname{sen} \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

$$\sqrt{(y')^2+1} = \sqrt{\operatorname{sen}^2 \frac{1}{x} + \frac{1}{x^2} \cos^2 \frac{1}{x} - \frac{1}{x} \operatorname{sen} \frac{2}{x} + 1}$$

Se $\frac{1}{x} \in \left[2k\pi - \frac{\pi}{4}, 2k\pi\right]$, então

$$\frac{2}{x} \in \left[4k\pi - \frac{\pi}{2}, 4\pi k\right] \quad (k \in \mathbb{N})$$

e temos que $\cos \frac{1}{x} \geq \frac{1}{\sqrt{2}}$ e

$\operatorname{sen} \frac{2}{x} \leq 0$

Assim,

$$\sqrt{(y')^2+1} \geq \sqrt{\frac{1}{x^2} \cdot \frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{x}$$

se

$$2k\pi - \frac{\pi}{4} \leq \frac{1}{x} \leq 2k\pi,$$

ou seja,

$$\frac{4}{8k\pi} \leq x \leq \frac{4}{(8k-1)\pi}$$

Temos que, $\forall\, k \in \mathbb{N}$,

$$S = \int_0^1 \sqrt{(y')^2 + 1}\, dx$$

$$\geq \sum_{k=1}^{\infty} \int_{\frac{4}{8k\pi}}^{\frac{4}{(8k-1)\pi}} \frac{1}{\sqrt{2}} \cdot \frac{1}{x}\, dx$$

$$\geq \sum_{k=1}^{\infty} \frac{1}{\sqrt{2}} \cdot \frac{(8k-1)\pi}{4} \cdot \left(\frac{4}{(8k-1)\pi} - \frac{4}{8k\pi} \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{2}} \cdot \frac{\cancel{(8k-1)\pi}}{\cancel{4}} \cdot \frac{\cancel{4}\cancel{\pi}}{\cancel{(8k-1)\pi} \cdot 8k\cancel{\pi}}$$

$$= \frac{1}{8\sqrt{2}} \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

Logo, a função $f(x) = x \operatorname{sen} \frac{1}{x}$ não é retificável em $[0, 1]$.

Exercício 1. Calcule a área das superfícies de revolução obtidas pela rotação ao redor do eixo x:

(a) $y = x^3$, $x \in [0,2]$ (b) $y = \sqrt{5-x}$, $x \in [3,5]$ (c) $y^2 = x+1$, $x \in [0,3]$

(d) $9x = y^2 + 18$, $x \in [2,6]$ (e) $y = \cos(x/2)$, $x \in [0,\pi]$ (f) $y = \operatorname{sen} \pi x$, $x \in [0,1]$

(a) $A = \int_0^2 2\pi x^3 \sqrt{9x^4+1}\, dx$

$= \frac{\pi}{27} \int_0^2 54x^3\sqrt{9x^4+1}\, dx$

$= \frac{\pi}{27} (9x^4+1)^{3/2} \Big|_0^2 = \frac{\pi}{27}\left[(145)^{3/2} - 1\right]$

$\frac{d}{dx}(9x^4+1)^{3/2} = \underbrace{\frac{3}{2}.36}_{54} x^3 \sqrt{9x^4+1}$

(b) $A = \int_3^5 2\pi\sqrt{5-x} \cdot \sqrt{\frac{1}{4(5-x)} + 1}\, dx$

$= \int_3^5 \cancel{2}\pi \cancel{\sqrt{5-x}}\, \frac{\sqrt{21-4x}}{\cancel{2\sqrt{5-x}}}\, dx$

$= -\frac{\pi}{6}(21-4x)^{3/2}\Big|_3^5$

$= \frac{\pi}{6}(27-1) = \frac{13\pi}{3}$

$\frac{d}{dx}(21-4x)^{3/2} = -6(21-4x)^{1/2}$

(c) $A = \int_0^3 2\pi\sqrt{x+1}\sqrt{\frac{1}{4(x+1)}+1}\,dx$

$= \int_0^3 \cancel{2}\pi\cancel{\sqrt{x+1}}\,\frac{\sqrt{4x+5}}{\cancel{2}\cancel{\sqrt{x+1}}}\,dx$

$\frac{d}{dx}(4x+5)^{3/2}$
$= 6(4x+5)^{1/2}$

$= \frac{\pi}{6}(4x+5)^{3/2}\Big|_0^3$

$= \frac{\pi}{6}\left[(17)^{3/2} - (5)^{3/2}\right]$

(d) $y = 3\sqrt{x-2}$

$A = \int_2^6 6\pi\sqrt{x-2}\cdot\sqrt{\frac{1}{4(x-2)}+1}\,dx$

$= \int_2^6 \cancel{6}^{3}\pi\cancel{\sqrt{x-2}}\,\frac{\sqrt{4x-7}}{\cancel{2}\cancel{\sqrt{x-2}}}\,dx$

$\frac{d}{dx}(4x-7)^{3/2}$
$= 6(4x-7)^{1/2}$

$= \frac{\pi}{2}(4x-7)^{3/2}\Big|_2^6$

$= \frac{\pi}{2}\left(17^{3/2} - 1\right)$

(e) $A = \displaystyle\int_0^{\pi} 2\pi \cos\left(\frac{x}{2}\right) \cdot \sqrt{\frac{1}{4}\operatorname{sen}^2\left(\frac{x}{2}\right) + 1}\; dx$

$u = \frac{x}{2} \Rightarrow 2\,du = dx$

$$= \overset{2}{\cancel{4}}\pi \int_0^{\frac{\pi}{2}} \cos u \sqrt{\frac{\operatorname{sen}^2 u + 4}{\underset{2}{\cancel{4}}}}\; du$$

$v = \operatorname{sen} u \Rightarrow dv = \cos u\, du$

$$= 2\pi \int_0^1 \sqrt{v^2 + 4}\; dv$$

$v = 2\operatorname{senh} w \Rightarrow dv = 2\cosh w\, dw$

$$= 8\pi \int_0^{\operatorname{arsenh} 1} \cosh^2 w\; dw$$

$\cosh 2w = \cosh^2 w + \operatorname{senh}^2 w = 2\cosh^2 w + 1$

$\Rightarrow \cosh^2 w = \dfrac{\cosh 2w - 1}{2}$

$$= \overset{4}{\cancel{8}}\pi \int_0^{\operatorname{arsenh} 1} \frac{\cosh 2w - 1}{\cancel{2}}\; dw$$

$$= 4\pi \cdot \left(\frac{\operatorname{senh} 2w}{2} - w\right)\Bigg|_0^{\operatorname{arsenh} 1}$$

$\operatorname{senh} w = 1 \Rightarrow \dfrac{e^{2w} - 1}{2e^w} = 1 \Rightarrow (e^w)^2 - 2e^w - 1 = 0$

$\Rightarrow e^w = 1 + \sqrt{2} \Rightarrow w = \log(1 + \sqrt{2}) = \operatorname{arsenh} 1$

$$= 4\pi\left(\operatorname{senh} w \cdot \cosh w - w\right)\Big|_0^{\operatorname{arsenh} 1}$$

$\operatorname{senh} w = 1 \Rightarrow \cosh^2 w = 1+1 \Rightarrow \cosh w = \sqrt{2}$

$$= 4\pi\left(\sqrt{2} - \log(1+\sqrt{2})\right)$$

(d) $$A = \int_0^1 2\pi \operatorname{sen} \pi x \sqrt{\pi^2 \cos^2 \pi x + 1}\, dx$$

$u = \pi \cos \pi x \Rightarrow du = -\pi^2 \operatorname{sen} \pi x\, dx$
$\Rightarrow \pi \operatorname{sen} \pi x\, dx = -\frac{du}{\pi}$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} \sqrt{u^2+1}\, du$$

$u = \operatorname{senh} w \Rightarrow du = \cosh w\, dw$

$$= \frac{2}{\pi}\int_{\operatorname{arsenh}(-\pi)}^{\operatorname{arsenh}\pi} \cosh^2 w\, dw$$

$\cosh 2w = \cosh^2 w + \operatorname{senh}^2 w$
$= 2\cosh^2 w - 1$

$$= \frac{1}{\pi}\int_{\operatorname{arsenh}(-\pi)}^{\operatorname{arsenh}\pi} \cosh 2w + 1\, dw = \frac{1}{\pi}\left(\frac{\operatorname{senh} 2w}{2} + w\right)\Bigg|_{\operatorname{arsenh}(-\pi)}^{\operatorname{arsenh}\pi}$$

$$= \frac{1}{\pi}\left(\operatorname{senh} w \cdot \cosh w + w\right)\Bigg|_{\operatorname{arsenh}(-\pi)}^{\operatorname{arsenh}\pi}$$

- $\operatorname{senh} w = \pi \Rightarrow \dfrac{e^{2w}-1}{2e^{w}} = \pi \Rightarrow e^{2w} - 2\pi e^{w} - 1 = 0$

$$\Rightarrow e^{w} = \frac{2\pi + \sqrt{4\pi^2+4}}{2} = \pi + \sqrt{\pi^2+1} = \operatorname{arsenh}\pi$$

- senh é ímpar $\Rightarrow \operatorname{arsenh}(-\pi) = -\left(\pi + \sqrt{\pi^2+1}\right)$
- $\operatorname{senh} w = \pm\pi \Rightarrow \cosh w = \sqrt{\pi^2+1}$

$$= \frac{1}{\pi}\left[2\pi\sqrt{\pi^2+1} + 2\left(\pi + \sqrt{\pi^2+1}\right)\right]$$

$$= 2\left(\sqrt{\pi^2+1} + 1 + \sqrt{1+\frac{1}{\pi^2}}\right)$$

Exercício 2. Calcule a área das seguintes superfícies de revolução em torno do eixo y:

(a) $x^{2/3} + y^{2/3} = 1$, $y \in [0,1]$ (b) $y = \frac{1}{4}x^2 - \frac{1}{2}\log x$, $x \in [1,2]$

(a) $x^{2/3} = 1 - y^{2/3} \Rightarrow x = (1 - y^{2/3})^{3/2}$

$x'(y) = -(1 - y^{2/3})^{1/2} \cdot y^{-1/3}$

$$A = \int_0^1 2\pi (1 - y^{2/3})^{3/2} \sqrt{\frac{1 - y^{2/3}}{y^{2/3}} + 1}\, dy$$

$$= \int_0^1 2\pi (1 - y^{2/3})^{3/2} \cdot y^{-1/3} dy$$

$$\frac{d}{dy}(1 - y^{2/3})^{5/2} = -\frac{5}{2} \cdot \frac{2}{3} \cdot y^{-1/3} \cdot (1 - y^{2/3})^{3/2} = -\frac{5}{3} y^{-1/3} (1 - y^{2/3})^{3/2}$$

$$= -\frac{6\pi}{5} (1 - y^{2/3})^{5/2} \Big|_0^1 = \frac{6\pi}{5}$$

(b) $$A = \int_1^2 2\pi x \sqrt{\frac{1}{4}\left(x - \frac{1}{x}\right)^2 + 1}\, dx$$

$$= \int_1^2 \cancel{2}\pi \cancel{x} \sqrt{\frac{(x^2-1)^2 + 4x^2}{\cancel{2x}}}\, dx$$

$$= \int_1^2 \pi \sqrt{x^4 - \cancel{2x^2} + 1 + \cancel{4}_{2}x^2}\, dx$$

$$= \pi \int_1^2 \sqrt{(x^2+1)^2}\, dx = \pi \int_1^2 x^2 + 1\, dx$$

$$= \pi \left(\frac{x^3}{3} + x \right) \Big|_1^2 = \pi \left(\frac{8}{3} + 2 - \frac{1}{3} - 1 \right)$$

$$= \frac{10\pi}{3}$$

Exercício 3. Se a curva infinita $y = e^{-x}$, $x \geqslant 0$ é girada em torno do eixo x, calcule a área da superfície resultante.

$$A = \int 2\pi y \, ds = \int_0^{\infty} 2\pi e^{-x} \sqrt{e^{-2x} + 1} \, dx$$

$$= 2\pi \int_0^{\infty} e^{-x} \frac{\sqrt{e^{2x}+1}}{e^{x}} \, dx$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx \Rightarrow dx = \frac{du}{2u}$$

$$= 2\pi \int_1^{\infty} \frac{\sqrt{u+1}}{u^2} \, du$$

$$v = \sqrt{u+1} \Rightarrow dv = \frac{1}{2\sqrt{u+1}} du$$

$$\Rightarrow du = 2v\,dv \, , \quad u = v^2 - 1$$

$$= 4\pi \int_{\sqrt{2}}^{\infty} \frac{v^2}{(v^2-1)^2} \, dv$$

$$= \pi \int_{\sqrt{2}}^{\infty} \frac{v^2 - 1 \quad + 1}{(v^2-1)^2} \, dv$$

$$= \pi \int_{\sqrt{2}}^{\infty} \frac{1}{v^2-1} + \frac{1}{(v^2-1)^2} \, dv$$

$$= \pi \int_{\sqrt{2}}^{\infty} \frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right) + \frac{1}{(v-1)^2(v+1)^2} \, dv$$

$$\frac{1}{(v-1)^2(v+1)^2} = \frac{a}{v-1} + \frac{b}{(v-1)^2} + \frac{c}{v+1} + \frac{d}{(v+1)^2}$$

$$\Rightarrow \frac{a(v^2-1)(v+1) + b(v+1)^2 + c(v^2-1)(v-1) + d(v-1)^2}{(v^2-1)^2}$$

$$= \left\{ a(v^3+v^2-v-1) + b(v^2+2v+1) + c(v^3-v^2-v+1) + d(v^2-2v+1) \right\} / (v^2-1)^2$$

$$\Rightarrow \begin{cases} a+c = 0 \Rightarrow c = -a \\ a+b-c+d = 0 \Rightarrow 2a+b+d = 0 \Rightarrow b = -2a-d \\ -a+2b-c-2d = 0 \Rightarrow -4a-4d = 0 \Rightarrow \begin{cases} d = -a \\ b = -a \end{cases} \\ -a+b+c+d = 1 \Rightarrow -4a = 1 \end{cases}$$

$$\Rightarrow a = \frac{-1}{4}, \quad b = c = d = \frac{1}{4}$$

Assim,

$$A = \pi \int_{\sqrt{2}}^{\infty} \frac{1}{2}\left(\frac{1}{v-1} - \frac{1}{v+1}\right) - \frac{1}{4}\frac{1}{v-1} + \frac{1}{4}\frac{1}{(v-1)^2} + \frac{1}{4}\frac{1}{v+1} + \frac{1}{4}\frac{1}{(v+1)^2}\,dv$$

$$= \pi \int_{\sqrt{2}}^{\infty} \frac{1}{4}\frac{1}{v-1} - \frac{1}{4}\frac{1}{v+1} + \frac{1}{4}\frac{1}{(v-1)^2} + \frac{1}{4}\frac{1}{(v+1)^2}\,dv$$

$$= \frac{\pi}{4}\left[\log\frac{v+1}{v-1} - \left(\frac{1}{v-1} + \frac{1}{v+1}\right)\right]\Bigg|_{\sqrt{2}}^{\infty}$$

$$= \frac{\pi}{4}\left(\log\frac{v+1}{v-1} - \frac{2v}{v^2-1}\right)\Bigg|_{\sqrt{2}}^{\infty}$$

$$= \frac{\pi}{4}\left(\frac{2\sqrt{2}}{2-1} - \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \frac{\pi}{4}\left(2\sqrt{2} - \log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Exercício 4. Encontre a área da superfície de revolução obtida a partir da elipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b$$

em cada caso:

(a) giro em torno do eixo x (elipsoide ou esferoide prolato)

(b) giro em torno do eixo y (esferoide oblato)

(a) $y = \frac{b}{a}\sqrt{a^2 - x^2}$

$\Rightarrow y' = -\frac{b}{a} \cdot \frac{x}{\sqrt{a^2-x^2}}$

$A = \int 2\pi y \, ds$

$= \int_{-a}^{a} 2\pi \frac{b}{a}\sqrt{a^2-x^2} \sqrt{\frac{b^2x^2}{a^2(a^2-x^2)} + \frac{a^4 - a^2x^2}{a^2(a^2-x^2)}} \, dx$

$= \int_{-a}^{a} 2\pi \frac{b}{a} \cancel{\sqrt{a^2-x^2}} \frac{\sqrt{a^4 - (a^2-b^2)x^2}}{a\cancel{\sqrt{a^2-x^2}}} \, dx$

$= \int_{-a}^{a} 2\pi b \sqrt{1 - \frac{a^2-b^2}{a^4}x^2} \, dx$

$\frac{\sqrt{a^2-b^2}}{a^2}x = \operatorname{sen} u \Rightarrow dx = \frac{a^2}{\sqrt{a^2-b^2}} \cos u \, du$

$$= \int_{\arcsin\left(-\frac{\sqrt{a^2-b^2}}{a}\right)}^{\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)} \frac{2\pi b a^2}{\sqrt{a^2-b^2}} \cos^2 u \, du$$

$$2\cos^2 u - 1 = \cos^2 u - \operatorname{sen}^2 u = \cos 2u$$
$$\Rightarrow 2\cos^2 u = \cos 2u + 1$$

$$= \frac{\pi b a^2}{\sqrt{a^2-b^2}} \int_{\arcsin\left(-\frac{\sqrt{a^2-b^2}}{a}\right)}^{\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)} \cos 2u + 1 \, du$$

$$= \frac{\pi b a^2}{\sqrt{a^2-b^2}} \left(\frac{\operatorname{sen} 2u}{2} + u\right) \Bigg|_{\arcsin\left(-\frac{\sqrt{a^2-b^2}}{a}\right)}^{\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)}$$

$$= \frac{\pi b a^2}{\sqrt{a^2-b^2}} \left(\operatorname{sen} u \cos u + u\right) \Bigg|_{\arcsin\left(-\frac{\sqrt{a^2-b^2}}{a}\right)}^{\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)}$$

$$\operatorname{sen} u = \pm \frac{\sqrt{a^2-b^2}}{a} \Rightarrow \cos u = \sqrt{\frac{a^2 - a^2 + b^2}{a^2}} = \frac{b}{a}$$

$$= \frac{\pi b a^2}{\sqrt{a^2-b^2}} \left[2\,\frac{\sqrt{a^2-b^2} \cdot b}{a^2} + 2 \arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]$$

$$= 2\pi\left[b^2 + \frac{ba^2}{\sqrt{a^2-b^2}}\,\text{arc sen}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]$$

(b) $x = \frac{a}{b}\sqrt{b^2-y^2}$

$$\Rightarrow x' = -\frac{a}{b}\cdot\frac{y}{\sqrt{b^2-y^2}}$$

$$A = \int 2\pi x\,ds$$

$$= \int_{-b}^{b} 2\pi\frac{a}{b}\sqrt{b^2-y^2}\sqrt{\frac{a^2y^2}{b^2(b^2-y^2)} + \frac{b^4-b^2y^2}{b^2(b^2-y^2)}}\,dy$$

$$= \int_{-b}^{b} 2\pi\frac{a}{b}\cancel{\sqrt{b^2-y^2}}\,\frac{\sqrt{b^4+(a^2-b^2)y^2}}{b\cancel{\sqrt{b^2-y^2}}}\,dx$$

$$= \int_{-b}^{b} 2\pi a\sqrt{1+\frac{a^2-b^2}{b^4}y^2}\,dy$$

$$\frac{\sqrt{a^2-b^2}}{b^2}y = \operatorname{senh} u \Rightarrow dy = \frac{b^2}{\sqrt{a^2-b^2}}\cosh u\,du$$

$$= \int_{\operatorname{arsenh}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}^{\operatorname{arsenh}\left(\frac{\sqrt{a^2-b^2}}{b}\right)} \frac{2\pi a b^2}{\sqrt{a^2-b^2}} \cosh^2 u \, du$$

$$\cosh 2u = \cosh^2 u + \operatorname{senh}^2 u = 2\cosh^2 u - 1$$

$$\Rightarrow 2\cosh^2 u = \cosh 2u + 1$$

$$= \int_{\operatorname{arsenh}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}^{\operatorname{arsenh}\left(\frac{\sqrt{a^2-b^2}}{b}\right)} \frac{\pi a b^2}{\sqrt{a^2-b^2}} (\cosh 2u + 1) \, du$$

$$= \frac{\pi a b^2}{\sqrt{a^2-b^2}} \left(\frac{\operatorname{senh} 2u}{2} + u \right) \Bigg|_{\operatorname{arsenh}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}^{\operatorname{arsenh}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{\pi a b^2}{\sqrt{a^2-b^2}} \left(\operatorname{senh} u \cosh u + u \right) \Bigg|_{\operatorname{arsenh}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}^{\operatorname{arsenh}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}$$

$$\operatorname{senh} u = \pm \frac{\sqrt{a^2-b^2}}{b} \Rightarrow \cosh u = \sqrt{1 + \frac{a^2-b^2}{b^2}} = \frac{a}{b}$$

$$= \frac{2\pi a b^2}{\sqrt{a^2-b^2}}\left(\frac{a\sqrt{a^2-b^2}}{b^2} + \operatorname{arsenh}\frac{\sqrt{a^2-b^2}}{b}\right)$$

$$\operatorname{senh} u = w \Rightarrow \frac{e^{2u}-1}{2e^u} = w \Rightarrow e^{2u} - 2we^u - 1 = 0$$

$$\Rightarrow e^u = \frac{2w + \sqrt{4w^2+4}}{2} = w + \sqrt{w^2+1}$$

$$\Rightarrow u = \operatorname{arsenh} w = \log\left(w + \sqrt{w^2+1}\right)$$

$$= 2\pi\left[a^2 + \frac{ab^2}{\sqrt{a^2-b^2}}\log\left(\frac{\sqrt{a^2-b^2}}{b} + \sqrt{\frac{a^2-b^2+b^2}{b}}\right)\right]$$

$$= 2\pi\left[a^2 + \frac{ab^2}{\sqrt{a^2-b^2}}\log\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)\right]$$

Exercício 5. Encontre a área da superfície do toro obtido pela rotação do círculo $(x-R)^2 + y^2 = r^2$ em torno do eixo y $(R > r)$.

$$A = \int 2\pi x \, ds$$

$$(x-R)^2 = r^2 - y^2$$

$$\Rightarrow x = R \pm \sqrt{r^2-y^2}, \qquad -r \leq y \leq r$$

$$\frac{dx}{dy} = \pm \frac{y}{\sqrt{r^2-y^2}}$$

$$\Rightarrow ds = \sqrt{\frac{y^2 + r^2 - y^2}{\sqrt{r^2-y^2}}}\, dy = \frac{r}{\sqrt{r^2-y^2}}\, dy$$

São duas funções. A área total é a soma das áreas.

$$A^{\pm} = \int_{-r}^{r} 2\pi\left(R \pm \sqrt{r^2-y^2}\right)\frac{r}{\sqrt{r^2-y^2}}\, dy$$

$$= \int_{-r}^{r} \frac{2\pi R r}{\sqrt{r^2-y^2}}\, dy \pm \int_{-r}^{r} 2\pi r \, dy \qquad \left\{ \begin{array}{l} u = \frac{y}{r} \\ r\,du = dy \end{array} \right.$$

$$= \int_{-1}^{1} \frac{2\pi R r}{\sqrt{1-y^2}}\, dy \pm 2\pi r y \Big|_{-r}^{r}$$

$$= 2\pi R r \cdot \arcsen y \Big|_{-1}^{1} \pm 4\pi r^2$$

$$= 4\pi^2 Rr \pm 4\pi r^2$$

Assim,

$$A = A^+ + A^- = 8\pi^2 Rr$$

Exercício 6. Mostre que, se girarmos a curva $y = e^{x/2} + e^{-x/2}$ em torno do eixo x, a área da superfície resultante tem o mesmo valor que o volume englobado, para qualquer intervalo $a \leqslant x \leqslant b$.

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{x/2} - e^{-x/2}\right) = \operatorname{senh}\frac{x}{2}$$

$$y = 2 \cdot \cosh\frac{x}{2}$$

Assim,

$$A = \int_a^b 2\pi y\, ds = \int_a^b 4\pi \cosh\frac{x}{2} \cdot \sqrt{\operatorname{senh}^2\frac{x}{2} + 1}\, dx$$

$$= \int_a^b 4\pi \cosh^2\frac{x}{2}\, dx$$

Por outro lado,

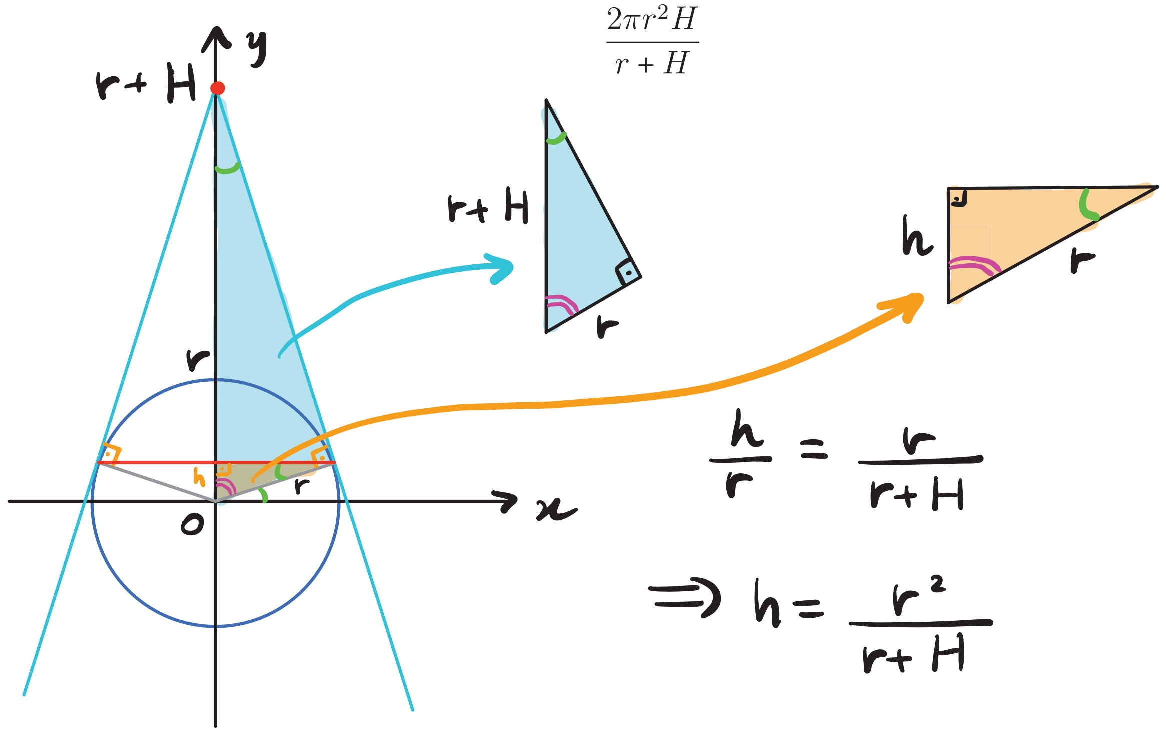
$$V = \int_a^b \pi y^2\, dx = \int_a^b 4\pi \cosh^2\frac{x}{2}\, dx$$

Assim,

$$V = A$$

Exercício 7. Mostre que um observador a uma altura H acima do polo norte de uma esfera de raio r pode ver uma parte da esfera que tem área

$$\frac{2\pi r^2 H}{r+H}$$



$$\frac{h}{r} = \frac{r}{r+H}$$

$$\Rightarrow h = \frac{r^2}{r+H}$$

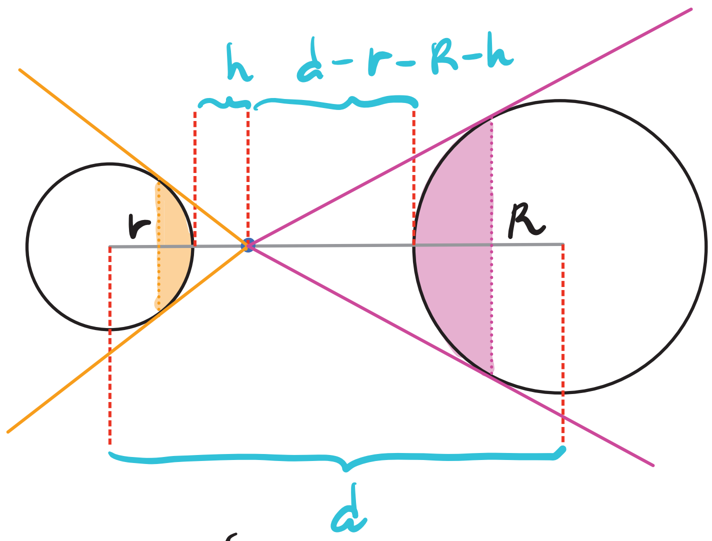
Temos que calcular a área da calota esférica em $\frac{r^2}{r+H} \leq y \leq r$, que é gerada pela revolução de $x = \sqrt{r^2-y^2}$ em torno do eixo y.

$$A = \int 2\pi x \, ds = \int_{r^2/(r+H)}^{r} 2\pi \sqrt{r^2-y^2} \sqrt{\frac{y^2}{r^2-y^2}+1} \, dy$$

$$= \int_{r^2/(r+H)}^{r} 2\pi \cancel{\sqrt{r^2-y^2}} \frac{r}{\cancel{\sqrt{r^2-y^2}}} \, dy = 2\pi r y \Big|_{r^2/(r+H)}^{r}$$

$$= 2\pi r^2 \left(1 - \frac{r}{r+H}\right) = \frac{2\pi r^2 H}{r+H}$$

Exercício 8. Duas esferas com raios r e R estão colocadas de modo que a distância entre os seus centros é d, onde $d > r + R$. Onde deve ser colocada uma luz na reta que liga os centros de modo a iluminar a maior área total de superfície?



Pelo exercício anterior, se a luz estiver a uma distância h da superfície da esfera de raio r (logo, a uma distância d-r-R-h da superfície da esfera de raio R), a área total iluminada é

$$A(h) = \frac{2\pi r^2 h}{r+h} + \frac{2\pi R^2(d-r-R-h)}{R+d-r-R-h}$$

$$= \frac{2\pi r^2[(r+h)-r]}{r+h} + \frac{2\pi R^2[(d-r-h)-R]}{d-r-h}$$

$$= 2\pi(r^2+R^2) - 2\pi\left[\frac{r^3}{r+h} + \frac{R^3}{d-r-h}\right]$$

$$\Rightarrow A'(h) = 2\pi\left[\frac{r^3}{(r+h)^2} - \frac{R^3}{(d-r-h)^2}\right] = 0$$

se

$$\frac{r^3}{(r+h)^2} = \frac{R^3}{[d-(r+h)]^2}$$

Seja $z = r+h$.

$$r^3(d-z)^2 = R^3 z^2$$

$$(R^3 - r^3)z^2 + 2dr^3 z - r^3 d^2 = 0$$

Temos 2 casos.

1) $R > r$ (ou $r < R$, caso prefira assim)

$$\Rightarrow z = -\frac{dr^3}{R^3 - r^3} \pm \frac{\sqrt{\cancel{4r^6d^2} + 4r^3R^3d^2 - \cancel{4r^6d^2}}}{2(R^3-r^3)}$$

$$= \frac{-dr^3 \pm d\sqrt{R^3r^3}}{R^3 - r^3}$$

$$\Rightarrow z = d\left(\frac{\sqrt{R^3r^3} - r^3}{R^3 - r^3}\right) \quad (\text{supondo } R > r)$$

Assim,

$$h = z - r = d\left(\frac{\sqrt{R^3r^3} - r^3}{R^3 - r^3}\right) - r.$$

2) $R = r$

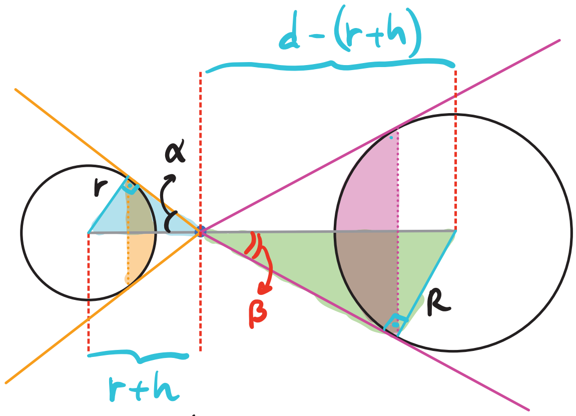
$$\Rightarrow z = \frac{r^3 d^2}{2r^3 d} = \frac{d}{2}$$

Em ambos os casos, o ponto crítico é de fato um máximo por conta do teste da 2ª derivada:

$$A''(h) = -4\pi\left[\frac{r^3}{(r+h)^3} + \frac{R^3}{(d-r-h)^3}\right] < 0$$

para $0 < h < d-r-R$.

OBS: Geometricamente, temos



A condição diz que

$$r\cdot\left(\frac{r}{r+h}\right)^2 = R\cdot\left[\frac{R}{d-(r+h)}\right]^2$$

$$\iff r\,\mathrm{sen}^2\alpha = R\,\mathrm{sen}^2\beta$$

$$\iff \sqrt{r}\,\mathrm{sen}\,\alpha = \sqrt{R}\,\mathrm{sen}\,\beta$$

Exercício 1. Usando a definição formal de limite, explique o que significa dizer que $\lim_{n\to\infty} a_n = 8$.

Significa dizer que, para qualquer $\varepsilon > 0$ arbitrário, é possível determinar $N_0 \in \mathbb{N}$ tal que se $n \geq N_0$, então

$$|a_n - 8| < \varepsilon.$$

Exercício 2. Usando a definição formal de limite, explique o que significa dizer que $\lim\limits_{n\to\infty} a_n = \infty$. E quanto a $\lim\limits_{n\to\infty} a_n = -\infty$?

Se $\lim\limits_{n\to\infty} a_n = \infty$, então $\forall M \in \mathbb{R}$ existe $N_0 \in \mathbb{N}$ tal que se $n \geq N_0$ então $a_n > M$.

Se $\lim\limits_{n\to\infty} a_n = -\infty$, então $\forall M \in \mathbb{R}$ existe $N_0 \in \mathbb{N}$ tal que se $n \geq N_0$ então $a_n < M$.

Exercício 3. Determine se a sequência converge ou diverge. Caso convirja, encontre o limite.

(a) $a_n = \dfrac{3n}{1+6n}$ (b) $a_n = 2 + \dfrac{(-1)^n}{n}$ (c) $a_n = 1 + \left(-\dfrac{1}{2}\right)^n$

(d) $a_n = 1 + \dfrac{10^n}{9^n}$ (e) $a_n = \dfrac{4n^2 - 3n}{2n^2+1}$ (f) $a_n = \dfrac{\log n}{\log 2n}$

(g) $a_n = \operatorname{sen} n$ (h) $a_n = \left(1 + \dfrac{2}{n}\right)^n$ (i) $a_n = \sqrt[n]{n}$

(j) $a_n = 2^{-n}\cos n\pi$ (k) $a_n = \arctan(\log n)$ (l) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

(m) $a_n = n\operatorname{sen}(1/n)$ (n) $a_n = \dfrac{n!}{2^n}$

(a) $\dfrac{3n}{1+6n} = \dfrac{3}{\frac{1}{n}+6} \xrightarrow[n\to\infty]{} \dfrac{1}{2}$

(b) $2 + \dfrac{(-1)^n}{n} \xrightarrow[n\to\infty]{} 2$

(c) $1 + \left(-\dfrac{1}{2}\right)^n \xrightarrow[n\to\infty]{} 1$

(d) $1 + \left(\dfrac{10}{9}\right)^n \xrightarrow[n\to\infty]{} \infty$ (diverge)

(e) $\dfrac{4n^2 - 3n}{2n^2+1} = \dfrac{4 - 3/n}{2 + 1/n^2} \xrightarrow[n\to\infty]{} 2$

(f) $\dfrac{\log n}{\log 2n} = \dfrac{\log n}{2 + \log n}$

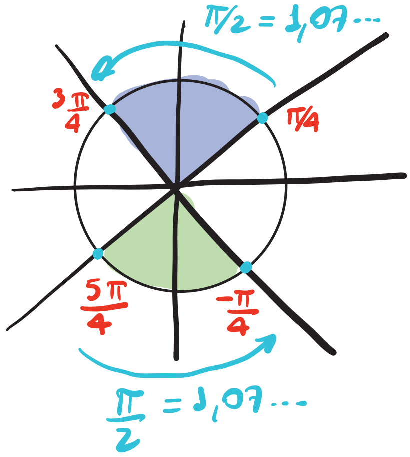
$= \dfrac{1}{\frac{2}{\log n} + 1} \xrightarrow[n\to\infty]{} 1$

(g) $a_n = \operatorname{sen} n$ diverge pois, para todo $k \in \mathbb{N}$, existem $p_k, q_k \in \mathbb{N}$ tais que

$$p_k \in \left(2k\pi + \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}\right)$$

$$q_k \in \left(2k\pi + \frac{5\pi}{4}, 2k\pi + \frac{7\pi}{4}\right),$$

visto que ambos os intervalos têm comprimento $\frac{\pi}{2} > 1$.



Assim, as subsequências

$$a_{p_k} = \operatorname{sen} p_k$$

e

$$a_{q_k} = \operatorname{sen} q_k$$

satisfazem

$$a_{p_k} > \operatorname{sen} \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \forall\, k \in \mathbb{N}$$

$$a_{q_k} < \operatorname{sen} -\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \quad \forall\, k \in \mathbb{N}$$

Logo, a_n não pode convergir.

(h) $\left(1+\frac{2}{n}\right)^n = \left[\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^2 \xrightarrow[n\to\infty]{} e^2$

(i) $\sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$

(j) $\frac{\cos n\pi}{2^n} \xrightarrow[n\to\infty]{} 0$

(k) $\arctan(\log n) \xrightarrow[n\to\infty]{} \frac{\pi}{2}$

(l) $n - \sqrt{n+1}\sqrt{n+3} =$

$$= \left(n-\sqrt{n+1}\sqrt{n+3}\right)\frac{\left(n+\sqrt{n+1}\sqrt{n+3}\right)}{n+\sqrt{n+1}\sqrt{n+3}}$$

$$= \frac{\cancel{n^2}-\cancel{n^2}-4n-3}{n+\sqrt{n+1}\sqrt{n+3}} = \frac{-4-3/n}{1+\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{3}{n}}}$$

$$\xrightarrow[n\to\infty]{} -4$$

(m) $n \operatorname{sen}\frac{1}{n} = \frac{\operatorname{sen} 1/n}{1/n} \xrightarrow[n\to\infty]{} 0$

(n) $\frac{n!}{2^n} \geq \frac{n\cdot 2^{n-1}}{2^n} = \frac{n}{2} \xrightarrow[n\to\infty]{} \infty$ (diverge)

Exercício 4. Determine se a sequência é crescente, decrescente ou não monótona. Diga também se ela é limitada.

(a) $a_n = \cos n$ (b) $a_n = n(-1)^n$ (c) $a_n = n^3 - 3n + 3$ (d) $a_n = \dfrac{1-n}{2+n}$

(a) Não monótona e limitada

$|a_n| \leq 1 \quad \forall\, n \in \mathbb{N}$

(b) Não monótona e ilimitada:

$a_{2n} \to +\infty, \quad a_{2n+1} \to -\infty$

(c) $f(x) = x^3 - 3x + 3$

$f'(x) = 3x^2 - 3 = 3(x^2-1) > 0$

se $x > 1$

Logo, $a_n = n^3 - 3n + 3$ é crescente, mas é ilimitada.

(d) $$\frac{1-n}{2+n} = \frac{-(2+n)+3}{2+n} = -1 + \frac{3}{2+n}$$

é decrescente. Temos $\lim_{n\to\infty} a_n = -1$.

Exercício 5. Mostre que a sequência definida por

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

é crescente e limitada. Deduza que ela é convergente e calcule o seu limite.

Considere

$$f(x) = 3 - \frac{1}{x} - x$$

Temos

$$f'(x) = \frac{1}{x^2} - 1 < 0 \quad \text{se } x > 1$$

Logo,

$$f(x) \text{ é crescente se } x > 1$$

$$\Rightarrow f(x) > f(1) = 3 - \frac{1}{1} - 1 = 1 > 0 \quad (x > 1)$$

$$\Rightarrow 3 - \frac{1}{x} > x \quad (\text{se } x \geq 1)$$

Assim,

$$a_{n+1} = 3 - \frac{1}{a_n} > a_n \quad \forall n \in \mathbb{N}$$

Logo, $(a_n)_{n \in \mathbb{N}}$ é crescente.

Por outro lado,

$$1 \leq a_n = 3 - \frac{1}{a_{n-1}} < 3 \quad \forall n \in \mathbb{N}.$$

Por ser monótona e limitada (a_n) converge. Seja $a = \lim a_n$. Então

$$a = 3 - \frac{1}{a} \Rightarrow a^2 - 3a + 1 = 0$$

$$\Rightarrow a = \frac{3 \pm \sqrt{5}}{2}$$

Como $a_n \geqslant 1 \; \forall n$, segue que

$$a = \frac{3+\sqrt{5}}{2}.$$

Exercício 6. Mostre que a sequência

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

é monótona e limitada. Conclua que converge e calcule o seu limite.

Considere

$$f(x) = \frac{1}{3-x} - x$$

Temos

$$f'(x) = \frac{1}{(3-x)^2} - 1 < 0 \quad \text{se} \quad x \geq 2$$

Logo,

$f(x)$ é decrescente se $x \geq 2$.

$$\Rightarrow f(x) < f(2) = \frac{1}{3-2} - 2 = -1 < 0$$

$$\Rightarrow \frac{1}{3-x} < x \quad (\text{se} \quad x \geq 2)$$

Assim,

$$a_{n+1} = \frac{1}{3-a_n} < a_n \quad \forall\, n \in \mathbb{N}$$

Logo, $(a_n)_{n\in\mathbb{N}}$ é decrescente.

Temos que $\quad a_n \leq a_1 = 2 < 3$

$$\Rightarrow 3 - a_n > 0 \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow a_{n+1} = \frac{1}{3-a_n} > 0$$

Assim, $(a_n)_{n\in\mathbb{N}}$ é monótona e limitada, portanto, converge. Seja

$a = \lim_{n\to\infty} a_n$. Então a satisfaz

$$a = \frac{1}{3-a} \Rightarrow 3a - a^2 = 1 \Rightarrow a^2 - 3a + 1 = 0$$

$$\Rightarrow a = \frac{3 \pm \sqrt{5}}{2}$$

Como $a_n \leq 2 \ \forall n$, segue que

$$a = \frac{3-\sqrt{5}}{2}$$

Exercício 7. Mostre pela definição de limite que $\lim_{n\to\infty} r^n = 0$ se $|r| < 1$.

Se $r=0$, então $\forall\ \varepsilon>0$, $|r|^n < \varepsilon$ para todo $n \geq 1$. Daí,

$$\lim_{n\to\infty} r^n = 0.$$

Assuma $0<|r|<1$. Seja $\varepsilon>0$.

Tome $N_0 \in \mathbb{N}$ tal que

$$N_0 > \frac{\log \varepsilon}{\log |r|}$$

Como $0<|r|<1$, $\log|r|<0$ e então temos que

$$N_0 \log|r| < \log \varepsilon$$

$$\Rightarrow |r|^{N_0} < \varepsilon$$

Como $|r|<1 \Rightarrow |r|^n < |r|^{n-1}$, segue que se $n \geq N_0$ então

$$|r^n| \leq |r|^{N_0} < \varepsilon$$

Daí, $\lim_{n\to\infty} r^n = 0$.

Exercício 8. Mostre que se $\lim_{n\to\infty} a_n = 0$ e $(b_n)_{n\in\mathbb{N}}$ for limitada, então $\lim_{n\to\infty}(a_n b_n) = 0$.

Seja $M > 0$ tal que $|b_n| < M$ $\forall n \in \mathbb{N}$, e seja $\varepsilon > 0$ arbitrário.

Como $\lim_{n\to\infty} a_n = 0$, existe $N_0 \in \mathbb{N}$ tal que se $n \geq N_0$ então

$$|a_n| < \frac{\varepsilon}{M}$$

$$\Rightarrow |a_n b_n| = |a_n| \cdot |b_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$$

se $n \geq N_0$.

Assim, $\lim_{n\to\infty} a_n b_n = 0$.

Exercício 9. Explique, usando a definição de limite, o que significa dizer que $\sum_{n=1}^{\infty} a_n = 5$.

Significa dizer que, dado $\varepsilon > 0$, existe $N_0 \in \mathbb{N}$ tal que se $N \geq N_0$ então

$$\left| \sum_{n=1}^{N} a_n - 5 \right| < \varepsilon$$

Exercício 10. Seja $a_n = \dfrac{2n}{3n+1}$.

(a) Determine se $(a_n)_{n\in\mathbb{N}}$ é convergente.

(b) Determine se $\sum_{n=1}^{\infty} a_n$ é convergente.

(a) $a_n = \dfrac{2}{3 + 1/n} \xrightarrow[n\to\infty]{} \dfrac{2}{3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n+1} \geq \sum_{n=1}^{\infty} \frac{2n}{3n+n} = \sum_{n=1}^{\infty} \frac{2}{4} = \infty$

Exercício 11. Determine se a série converge ou diverge. Caso convirja, calcule seu valor.

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n}\right)$ (b) $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$ (c) $\sum_{n=1}^{\infty}\left(e^{1/n}-e^{1/(n+1)}\right)$

(d) $\sum_{n=4}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (e) $\sum_{n=1}^{\infty}\log\frac{n}{n+1}$ (f) $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^n}$

(g) $\sum_{n=1}^{\infty}\frac{5}{\pi^n}$ (h) $\sum_{n=1}^{\infty}\frac{2+n}{1-2n}$ (i) $\sum_{n=1}^{\infty}\frac{1}{1+\left(\frac{2}{3}\right)^n}$

(j) $\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$ (k) $\sum_{n=1}^{\infty}\frac{e^n}{n^2}$

(a) $$\sum_{n=1}^{N}\frac{1}{n+2}-\frac{1}{n}=\sum_{n=3}^{N+2}\frac{1}{n}-\sum_{n=1}^{N}\frac{1}{n}$$

$$=-1-\frac{1}{2}+\frac{1}{N+1}+\frac{1}{N+2}\xrightarrow[N\to\infty]{}-\frac{3}{2}$$

(b) $$\sum_{n=1}^{N}\frac{3}{n(n+3)}=\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+3}\right)$$

$$=\sum_{n=1}^{N}\frac{1}{n}-\sum_{n=4}^{N+3}\frac{1}{n}$$

$$=1+\frac{1}{2}+\frac{1}{3}-\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{N+3}$$

$$\xrightarrow[N\to\infty]{}\frac{11}{6}$$

(c) $$\sum_{n=1}^{N}e^{1/n}-e^{1/(n+1)}=\sum_{n=1}^{N}e^{1/n}-\sum_{n=2}^{N+1}e^{1/n}$$

$$=e-e^{1/(N+1)}\xrightarrow[N\to\infty]{}e-1$$

(d) $$\sum_{n=4}^{N} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \sum_{n=4}^{N} \frac{1}{\sqrt{n}} - \sum_{n=5}^{N+1} \frac{1}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{N+1}} \xrightarrow[N\to\infty]{} \frac{1}{2}$$

(e) $$\sum_{n=1}^{N} \log \frac{n}{n+1} = \sum_{n=1}^{N} \log n - \log n+1$$

$$= \sum_{n=1}^{N} \log n - \sum_{n=2}^{N+1} \log n = -\log(N+1)$$

$$\xrightarrow[N\to\infty]{} -\infty$$

(f) $$\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} = \frac{1}{4} \; \frac{1}{1+\frac{3}{4}} = \frac{1}{7}$$

(g) $$\sum_{n=1}^{\infty} \frac{5}{\pi^n} = \frac{5}{\pi} \cdot \frac{1}{1-\frac{1}{\pi}} = \frac{5}{\pi - 1}$$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ diverge pois

$$\left|\frac{2+n}{1-2n}\right| = \left|\frac{1+2/n}{1/n-2}\right| \xrightarrow[n\to\infty]{} \frac{1}{2} \neq 0.$$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$ diverge pois

$$\lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = 1 \neq 0$$

(j) $$\sum_{n=1}^{\infty}\left[\frac{1}{e^n}+\frac{1}{n(n+1)}\right] = \sum_{n=1}^{\infty}\frac{1}{e^n} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$= \frac{1}{e}\,\frac{1}{1-\frac{1}{e}} + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \frac{1}{e-1} + 1$$

(k) $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^2}$ diverge pois

$$\lim_{n\to\infty}\frac{e^n}{n^2} = \infty$$

Exercício 12. Encontre os valores de x para os quais a série converge. Calcule a soma da série para esses valores de x.

(a) $\sum_{n=1}^{\infty}(-5)^n x^n$ (b) $\sum_{n=1}^{\infty}(x+2)^n$ (c) $\sum_{n=0}^{\infty}\frac{(9x-2)^n}{3^n}$ (d) $\sum_{n=0}^{\infty}e^{nx}$ (e) $\sum_{n=0}^{\infty}\frac{\operatorname{sen}^n x}{3^n}$

(a) A série converge se $|x|>5$:

$$\sum_{n=1}^{\infty}\left(-\frac{5}{x}\right)^n = \frac{1}{1+\frac{5}{x}} = \frac{x}{x+5}$$

(b) A série converge se $|x+2|<1$

$$\sum_{n=1}^{\infty}(x+2)^n = \frac{1}{1-(x+2)} = \frac{-1}{x+1}$$

(c) A série converge se $\left|\frac{9x-2}{3}\right|<1$

$$\iff |9x-2|<3 \iff \left|x-\frac{2}{9}\right|<\frac{1}{3}$$

$$\sum_{n=1}^{\infty}\left(\frac{9x-2}{3}\right)^n = \frac{1}{1-\frac{9x-2}{3}} = \frac{3}{5-9x}$$

(d) A série converge se $|e^x|<1$

$$\iff x<0$$

$$\sum_{n=0}^{\infty}e^{nx} = \frac{1}{1-e^x}$$

(e) A série converge $\forall x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \left(\frac{\operatorname{sen} x}{3}\right)^n = \frac{1}{1 - \frac{\operatorname{sen} x}{3}} = \frac{3}{3 - \operatorname{sen} x}$$

Exercício 13. Considere a série $\sum_{n=1}^{\infty} n/(n+1)!$. Calcule as primeiras somas parciais s_1, s_2, s_3, s_4. Conjecture uma fórmula para s_n e prove sua validade usando indução matemática. Em seguida, calcule o valor da soma da série.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{2}{3!} = \frac{1}{2} + \frac{1}{3} = \frac{5}{3!} = \frac{3!-1}{3!}$$

$$S_3 = \frac{5}{3!} + \frac{3}{4!} = \frac{23}{4!} = \frac{4!-1}{4!}$$

$$S_4 = \frac{4!-1}{4!} + \frac{4}{5!} = \frac{5!-5+4}{5!} = \frac{5!-1}{5!}$$

Conjectura: $S_n = \dfrac{(n+1)!-1}{(n+1)!}$

Assuma que valha para n como hipótese indutiva. Então:

$$S_{n+1} = S_n + \frac{n+1}{(n+2)!} = \frac{(n+1)!-1}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$$= \frac{(n+2)! - (n+2) + (n+1)}{(n+2)!} = \frac{(n+2)!-1}{(n+2)!}$$

Logo, a formula vale para S_{n+1}, completando a indução. Daí,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{(n+1)!-1}{(n+1)!}$$

$$= 1$$

Exercício 14. O Conjunto de Cantor é construído da seguinte forma. Começamos com o intervalo fechado $[0, 1]$ e removemos o intervalo aberto $\left(\frac{1}{3}, \frac{2}{3}\right)$. Isso nos leva a dois intervalos, $\left[0, \frac{1}{3}\right]$ e $\left[\frac{2}{3}, 1\right]$, e removemos cada terço aberto intermediário. Quatro intervalos permanecem, e novamente repetimos o processo. Continuamos esse procedimento indefinidamente, em cada passo removendo o terço aberto do meio de cada intervalo remanescente do passo anterior. O Conjunto de Cantor consiste de todos os números em $[0, 1]$ que permanecem depois de todos os infinitos intervalos terem sido removidos.

Mostre que o comprimento total de todos os intervalos que foram removidos é 1. Nesse sentido, dizemos que o Conjunto de Cantor tem medida nula. Apesar disso, o Conjunto de Cantor contém infinitos números. (Mais do que isso: ele é um exemplo de conjunto não enumerável de medida nula, ou seja, que não pode ser posto em uma correspondência bijetora com os números naturais).

Conjunto de Cantor

$C_1 = [0, 1]$

$C_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

$C_3 = \left[0, \frac{1}{3^2}\right] \cup \left[\frac{2}{3^2}, \frac{3}{3^2}\right] \cup \left[\frac{6}{3^2}, \frac{7}{3^2}\right] \cup \left[\frac{8}{3^2}, 1\right]$

$\vdots$

C_n é a união de 2^{n-1} intervalos de tamanho $\frac{1}{3^{n-1}}$

Medida do complementar $[0, 1] \setminus C_n$:

$$m_n = 1 - \frac{2^{n-1}}{3^{n-1}} \xrightarrow[n \to \infty]{} 1.$$

Exercício 15. Encontre a soma da série

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

(*Sugestão: mostre primeiro que* $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2\cot x$)

Primeiramente, note que

$$\cot x = \frac{\cos x}{\operatorname{sen} x} = \frac{\cos^2 \frac{x}{2} - \operatorname{sen}^2 \frac{x}{2}}{2 \operatorname{sen} \frac{x}{2} \cos \frac{x}{2}}$$

$$= \frac{1}{2}\left[\cot \frac{x}{2} - \tan \frac{x}{2}\right]$$

$$\Rightarrow 2\cot x = \cot \frac{x}{2} - \tan \frac{x}{2}$$

$$\Rightarrow \tan \frac{x}{2} = \cot \frac{x}{2} - 2\cot x$$

Para encontrar o valor da série, vamos considerar os seguintes casos:

Se $x = 0$:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 0 = 0$$

Se $x \neq 0$:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}\left[\cot \frac{x}{2^n} - 2\cot \frac{x}{2^{n-1}}\right]$$

$\tan \frac{x}{2^n} = T_n$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{T_n}{2^n} = \sum_{n=1}^{\infty} \left[\frac{1}{2^n} \cdot \frac{1}{T_n} - 2 \frac{1}{2^n} \cdot \frac{1}{T_{n-1}} \right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \cdot \frac{1}{T_n} - \frac{1}{2^{n-1}} \cdot \frac{1}{T_{n-1}} \right)$$

$$\frac{1}{T_n} = \frac{1}{\tan(x/2^N)} = \cot(x/2^N)$$

$$= \lim_{N\to\infty} \frac{\cot(x/2^N)}{2^N} - \cot x$$

O limite é indeterminado do tipo ∞/∞.

L'Hôpital:

$$\lim_{N\to\infty} \frac{\cot(x \cdot 2^{-N})}{2^N} =$$

$$= \lim_{N\to\infty} \frac{\cancel{-}\operatorname{cossec}^2(x \cdot 2^{-N}) \cdot x \cdot (\cancel{-\log 2})\, 2^{-N}}{2^N \cdot \cancel{\log 2}} \cdot \frac{x}{x}$$

$$= \lim_{N\to\infty} \left[\frac{x \cdot 2^{-N}}{\operatorname{sen}(x \cdot 2^{-N})} \right]^2 \cdot \frac{1}{x} = \frac{1}{x}$$

Logo, se $x \neq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{1}{x} - \cot x$$

Temos:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \begin{cases} \frac{1}{x} - \cot x & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Exercício 16. Encontre a soma da série

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\ldots$$

onde os termos são os recíprocos dos inteiros positivos cujos únicos fatores primos são 2 e 3.

Os inteiros cujos únicos fatores primos são 2 e 3 podem ser escritos como $2^n \cdot 3^m$, com $n, m \in \mathbb{N}^*$.

Como os termos são positivos, qualquer rearranjo dá a mesma soma. Logo,

$$S = \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\underbrace{\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)}_{\text{const. p/ } n}\right]$$

$$=\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right)\left(\sum_{n=0}^{\infty}\frac{1}{3^n}\right)$$

$$=\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}$$

$$=2\cdot\frac{3}{2}=3$$

Exercício 17. Encontre a soma da série $\sum_{n=2}^{\infty} \log\left(1 - \frac{1}{n^2}\right)$

Note que

$$\log\left(1-\frac{1}{n^2}\right) = \log\left(\frac{n^2-1}{n^2}\right) = \log\left[\frac{(n+1)}{n}\frac{(n-1)}{n}\right]$$

$$= \log(n+1) + \log(n-1) - 2\log n$$

Assim,

$$\sum_{n=2}^{N} \log\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{N}\left[\log(n+1) + \log(n-1) - 2\log n\right]$$

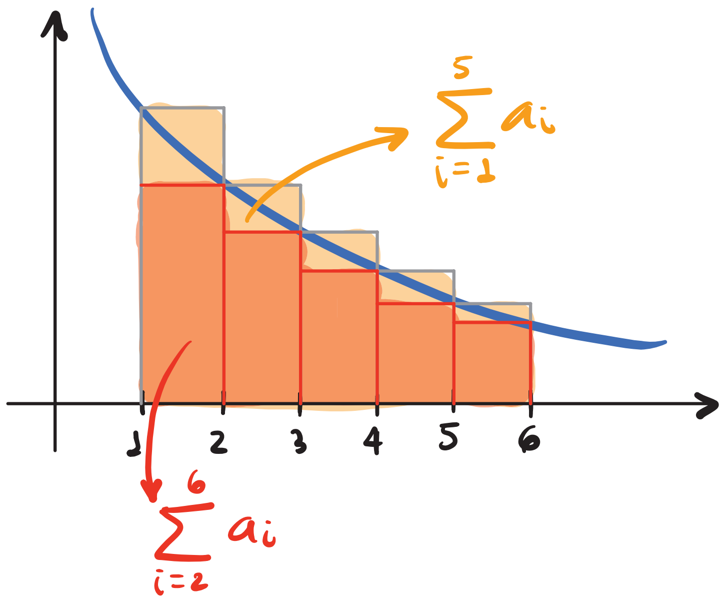
$$= \sum_{n=3}^{N+1} \log n + \sum_{n=1}^{N-1} \log n - 2\sum_{n=2}^{N} \log n$$

$$= \log(N+1) - \log 2 + \log 1 - \log N$$

$$= \log\left(\frac{N+1}{N}\right) - \log 2 \xrightarrow[N\to\infty]{} -\log 2$$

Exercício 1. Suponha que f seja contínua, positiva e decrescente para $x \geqslant 1$ e $a_n = f(n)$. Desenhando uma figura, coloque em ordem crescente as três quantidades:

$$\int_1^6 f(x)\,dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$



Temos $\displaystyle\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$,

pois

$$a_{i+1} = f(i+1) < \int_i^{i+1} f(x)\,dx < f(i) = a_i$$

$$\Rightarrow \sum_{i=1}^{5} a_{i+1} < \sum_{i=1}^{5} \int_i^{i+1} f(x)\,dx < \sum_{i=1}^{5} a_i$$

$$\Rightarrow \sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$$

Exercício 2. Use o Teste da Integral para determinar se a série é convergente ou divergente:

(a) $\sum_{n=1}^{\infty} n^{-3}$ (b) $\sum_{n=1}^{\infty} \frac{2}{5n-1}$ (c) $\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$ (d) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$

(e) $\sum_{n=1}^{\infty} n^{-0,3}$ (f) $\sum_{n=1}^{\infty} n^3 e^{-n^2}$ (g) $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$

(a) Converge pois

$$\int_1^{\infty} \frac{dx}{x^3} = \frac{-1}{2x^2}\Big|_1^{\infty} = \frac{1}{2}$$

(b) Diverge pois

$$\int_1^{\infty} \frac{2}{5x-1}\,dx = \frac{2}{5}\log(5x-1)\Big|_1^{\infty} = \infty$$

(c) Diverge pois

$$\int_2^{\infty} \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\log(x^3+1)\Big|_2^{\infty} = \infty$$

(d) Converge pois

$$\int_2^{\infty} \frac{dx}{x(\log x)^3} = \frac{-1}{2}\cdot\frac{1}{(\log x)^2}\Big|_2^{\infty} = \frac{1}{2(\log 2)^2}$$

(e) Diverge pois

$$\int_1^\infty \frac{1}{x^{0,3}}dx = \frac{1}{0,7} \cdot x^{0,7}\Big|_1^\infty = \infty$$

(f) Converge pois

$$\int_1^\infty x^3 e^{-x^2} dx = -\frac{1}{2}x^2 e^{-x^2}\Big|_1^\infty + \frac{1}{2}\int_1^\infty x^2 e^{-x^2} dx$$

$$= \frac{1}{2e} - \frac{1}{4} x e^{-x^2}\Big|_1^\infty + \frac{1}{4}\int_1^\infty x e^{-x^2} dx$$

$$= \frac{1}{2e} + \frac{1}{4e} - \frac{1}{8} e^{-x^2}\Big|_1^\infty = \frac{1}{2e} + \frac{1}{4e} + \frac{1}{8e} = \frac{7}{8e}$$

(g) Converge pois

$$\int_1^\infty \frac{\arctan x}{1+x^2} dx = \frac{1}{2}(\arctan x)^2 \Big|_1^\infty$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \frac{\pi^2}{2} \cdot \frac{3}{16} = \frac{3\pi^2}{32}$$

Exercício 3. Determine se a série converge ou diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ (b) $\sum_{n=3}^{\infty} n^{-0,9999}$ (c) $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots$

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$ (e) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ (f) $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \ldots$

(g) $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ (h) $\sum_{k=1}^{\infty} ke^{-k}$ (i) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \ldots$

(j) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

(a) Converge pois $\int_1^{\infty} \frac{1}{x^{\sqrt{2}}} dx = \frac{1}{1-\sqrt{2}} \left. \frac{1}{x^{\sqrt{2}-1}} \right|_1^{\infty}$

$= \frac{1}{\sqrt{2}-1}$

(b) Diverge pois $\int_3^{\infty} \frac{1}{x^{0,9999}} dx$

$= \frac{1}{0,0001} \cdot \left. x^{0,0001} \right|_1^{\infty} = \infty$

(c) Converge pois $\int_1^{\infty} \frac{1}{x^3} dx = -\frac{1}{2} \left. x^{-2} \right|_1^{\infty} = \frac{1}{2}$

(d) Converge pelo teste de comparação no limite pois

$$\frac{(\sqrt{n}+4)/n^2}{1/n^{3/2}} = \frac{\sqrt{n}+4}{\cancel{n^2}\sqrt{n}} \cdot \cancel{n^{3/2}} = 1 + \frac{4}{\sqrt{n}}$$

$\xrightarrow[n\to\infty]{} 1$ e $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converge pois

$$\int_1^\infty \frac{dx}{x^{3/2}} = -2x^{-1/2}\Big|_1^\infty = 2$$

(e) Diverge pois

$$\int_1^\infty \frac{dx}{x\log x} = \log(\log x)\Big|_1^\infty = \infty$$

(f) Diverge pois

$$\sum_{n=1}^\infty \frac{1}{4n-1} > \sum_{n=1}^\infty \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^\infty \frac{1}{n} = \infty$$

(g) Converge pois

$$\int_2^\infty \frac{\log x}{x^2}\,dx = -\frac{\log x}{x}\Big|_2^\infty + \int_2^\infty \frac{dx}{x^2}$$

$$= \frac{\log 2}{2} - \frac{1}{x}\Big|_2^\infty = \frac{\log 2}{2} - \frac{1}{2}$$

(h) Converge pois

$$\int_1^\infty x e^{-x}\,dx = -xe^{-x}\Big|_1^\infty + \int_1^\infty e^{-x}dx$$

$$= \frac{1}{e} - e^{-x}\Big|_1^\infty = \frac{2}{e}$$

(i) Converge pois

$$\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx = \int_1^{\infty} \frac{1}{x^{3/2}}\,dx = -2x^{-1/2}\Big|_1^{\infty} = 2$$

(j) Converge pelo critério de Leibniz pois

$$\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Exercício 4. Encontre os valores de p para os quais a série converge:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (b) $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n[\log(\log n)]^p]}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^p}$

(a) Pelo teste da integral, a série converge se e só se a seguinte integral converge:

$$\int_2^{\infty} \frac{1}{x(\log x)^p}\,dx = \begin{cases} \log(\log x)\Big|_2^{\infty} & \text{se } p = 1 \\ \dfrac{(\log x)^{1-p}}{1-p}\Big|_2^{\infty} & \text{se } p \neq 1 \end{cases}$$

Logo a convergência ocorre apenas se $p > 1$.

(b) Pelo teste da integral, a série converge se e só se a seguinte integral converge:

$$\int_1^{\infty} \frac{x}{(1+x^2)^{-p}}\,dx = \begin{cases} \dfrac{1}{2}\log(1+x^2)\Big|_1^{\infty} & \text{se } p = -1 \\ \dfrac{1}{2}\cdot\dfrac{(1+x^2)^{1+p}}{1+p}\Big|_1^{\infty} & \text{se } p \neq -1 \end{cases}$$

Logo a convergência ocorre apenas se $p < -1$.

(c) Pelo teste da integral, a série converge se e só se a seguinte integral converge:

$$\int_2^{\infty} \frac{dx}{x \log x (\log(\log x))^p} =$$

$$= \begin{cases} \log(\log(\log x)) \Big|_2^{\infty} & \text{se } p = 1 \\ \dfrac{[\log(\log x)]^{1-p}}{1-p} \Big|_2^{\infty} & \text{se } p \neq 1 \end{cases}$$

Logo, a convergência ocorre apenas se $p > 1$.

(d) Pelo teste da integral, a série converge se e só se a seguinte integral converge:

$$\int_1^{\infty} \frac{\log x}{x^p}\, dx = \frac{1}{1-p} \frac{\log x}{x^{p-1}} \Big|_1^{\infty} - \frac{1}{1-p} \int_1^{\infty} \frac{dx}{x^p}$$

(se $p \neq 1$)

$$= \frac{1}{1-p} \frac{\log x}{x^{p-1}} \Big|_1^{\infty} - \frac{1}{(1-p)^2} \frac{1}{x^{p-1}} \Big|_1^{\infty},$$

que só converge se $p > 1$. Quando $p = 1$, temos

$$\int_1^{\infty} \frac{\log x}{x} dx = \frac{1}{2} (\log x)^2 \Big|_1^{\infty} = \infty.$$

Logo, a convergência ocorre apenas se $p > 1$.

Exercício 5. (*Função Zeta de Riemann*) A função ζ é inicialmente definida por

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

onde $s \in \mathbb{C}$. Nesse formato, porém, ela não converge para todos os números complexos. Para quais números reais x a função $\zeta(x)$ (no formato acima) está definida?

Pelo teste da comparação da integral, $\zeta(x)$ converge se e só se a seguinte integral converge:

$$\int_1^\infty \frac{du}{u^x} = \begin{cases} \log u \Big|_1^\infty & \text{se } x = 1 \\ \dfrac{u^{1-x}}{1-x} \Big|_1^\infty & \text{se } x \neq 1 \end{cases}$$

Logo, $\zeta(x)$ converge somente se $x > 1$.

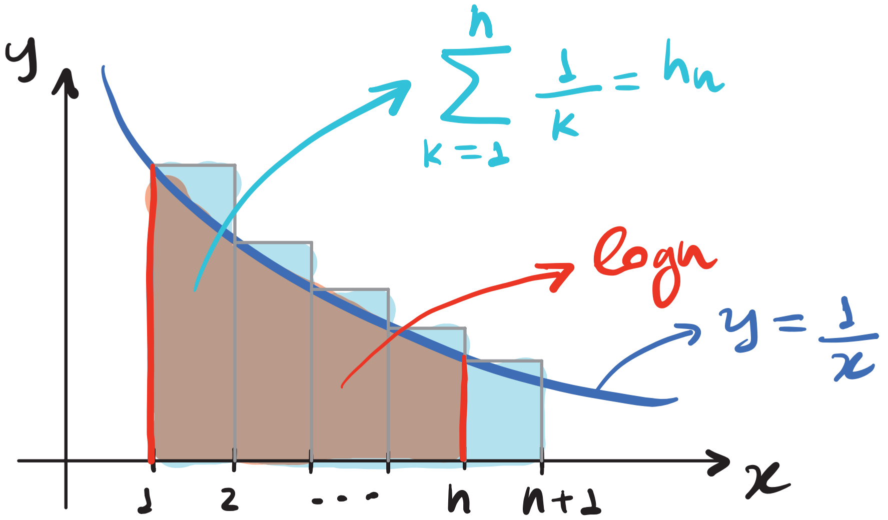
Exercício 6. (*Constante de Euler-Mascheroni*) Mostre que a sequência

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$$

converge. Seu limite é denotado como γ e é conhecido como constante de Euler-Mascheroni.

(*Sugestão: mostre primeiro que* $t_n > 0$ *ao comparar a série harmônica truncada com uma integral. Depois analise o comportamento de* $t_{n+1} - t_n$.)

Seja $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$



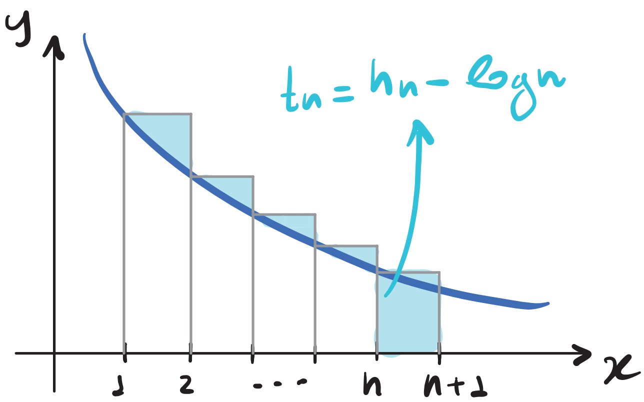
Temos que

$$\frac{1}{k} > \frac{1}{x} \quad \text{se} \quad k < x < k+1$$

$$\Rightarrow \frac{1}{k} > \int_k^{k+1} \frac{dx}{x}$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} > \int_1^n \frac{dx}{x}$$

$$\Rightarrow t_n = h_n - \log n > 0 \qquad (n > 1)$$



Temos que

$$t_{n+1} - t_n = \sum_{k=1}^{n+1} \frac{1}{k} - \log_{n+1} - \sum_{k=1}^{n} \frac{1}{k} + \log n$$

$$= \frac{1}{n+1} - (\log_{n+1} - \log n)$$

$$= \frac{1}{n+1} - \int_n^{n+1} \frac{dx}{x} = \int_n^{n+1} \left(\frac{1}{n+1} - \frac{1}{x}\right) dx < 0$$

pois $\frac{1}{x} > \frac{1}{n+1}$ em $[n, n+1)$.

Logo, $(t_n)_{n\in\mathbb{N}}$ é decrescente e limitada (pois $t_n > 0$). Portanto, como sequências monótonas e limitadas convergem, existe $\gamma \in \mathbb{R}$ tal que

$$\gamma = \lim_{n\to\infty} t_n$$

Exercício 7. Encontre todos os valores positivos de b para os quais a série $\sum_{n=1}^{\infty} b^{\log n}$ converge.

$$b^{\log n} = e^{\log b^{\log n}} = e^{(\log n)\log b}$$

$$= e^{\log(n^{\log b})} = n^{\log b}$$

Temos

$$\sum_{n=1}^{\infty} b^{\log n} = \sum_{n=1}^{\infty} n^{\log b},$$

que converge se e só se

$$\log b < -1 \iff 0 < b < \frac{1}{e}$$

Exercício 8. Encontre todos os valores de c para os quais a série converge:

$$\sum_{n=1}^{\infty}\left(\frac{c}{n}-\frac{1}{n+1}\right)$$

$$\frac{c}{n}-\frac{1}{n+1}=\frac{(c-1)n+c}{n(n+1)}=\frac{(c-1)n}{n(n+1)}+\frac{c}{n(n+1)}$$

Temos que $\sum \frac{c}{n(n+1)}$ converge $\forall c \in \mathbb{R}$.

Assim, se

$$\sum\left(\frac{c}{n}-\frac{1}{n+1}\right)=\sum\left[\frac{(c-1)n}{n(n+1)}+\frac{c}{n(n+1)}\right]$$

converge, então

$$\sum\frac{(c-1)n}{n(n+1)}=\sum\left[\frac{(c-1)n}{n(n+1)}+\frac{c}{n(n+1)}\right]-\sum\frac{c}{n(n+1)}$$

também converge.

Ocorre que, se $c \neq 1$,

$$\sum\frac{(c-1)n}{n(n+1)}=(c-1)\sum\frac{1}{n+1} \text{ diverge.}$$

Daí, a única possibilidade é

$$c=1.$$

Exercício 9. Verifique se as séries abaixo convergem:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^3+5}$ (b) $\displaystyle\sum_{n=2} \frac{n}{n^3-5}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log n}$

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$ (g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$ (j) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\operatorname{sen} n}{n^2}$ (k) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ (l) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

(m) $\displaystyle\sum_{n=1}^{\infty} \operatorname{sen}\left(\frac{1}{n}\right)$ (n) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\tan\frac{1}{n}$ (o) $\displaystyle\sum_{m=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ (p) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

(q) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\cos n}{n^3}$ (r) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ (s) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

(a) Converge pelo teste de comparação no limite, pois

$$\frac{n}{n^3+5} \Big/ \frac{1}{n^2} = \frac{n}{n^3+5}\cdot\frac{n^2}{1} = \frac{n^3}{n^3+5} = \frac{1}{1+5/n^3} \xrightarrow[n\to\infty]{} 1$$

e $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converge.

(b) Converge pelo teste de comparação no limite, pois

$$\frac{n}{n^3-5} \Big/ \frac{1}{n^2} = \frac{n}{n^3-5}\cdot\frac{n^2}{1} = \frac{n^3}{n^3-5} = \frac{1}{1-5/n^3} \xrightarrow[n\to\infty]{} 1$$

e $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converge.

(c) Converge pelo teste de comparação no limite, pois

$$\frac{1}{n^3+8} \Big/ \frac{1}{n^3} = \frac{n^3}{n^3+8} = \frac{1}{1+\frac{8}{n^3}} \xrightarrow[n\to\infty]{} 1$$

e $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converge.

(d) Diverge pois

$$n > \log n \Rightarrow \frac{1}{n} < \frac{1}{\log n}$$

e $$\sum_{n=2}^{\infty} \frac{1}{\log n} \geq \sum_{n=2}^{\infty} \frac{1}{n} = \infty.$$

(e) Converge pelo teste de comparação no limite, pois

$$\frac{\sqrt[3]{K}}{\sqrt{K^3+4K+3}} \Big/ \frac{1}{K^{7/6}} = \frac{K^{1/3} \cdot K^{7/6}}{(K^3+4K+3)^{1/2}} = \frac{K^{9/6}}{(K^3+4K+3)^{1/2}}$$

$$= \frac{1}{\left[\frac{1}{K^{9/3}} \cdot \left(K^3+4K+3\right)\right]^{1/2}} = \frac{1}{\left(1+\frac{4}{K^2}+\frac{3}{K^3}\right)^{1/2}}$$

$$\xrightarrow[k\to\infty]{} 1,$$

e $\sum_{k=1}^{\infty} \frac{1}{k^{7/6}}$ converge.

(f) Converge pois

$$\sum_{n=1}^{\infty} \left|\frac{1+\cos n}{e^n}\right| < \sum_{n=1}^{\infty} \frac{2}{e^n} = \frac{2}{e} \cdot \frac{1}{1-\frac{1}{e}}$$

$$= \frac{2}{e-1}$$

(g) Diverge pelo teste de comparação no limite, pois

$$\frac{1}{\sqrt{n^2+1}} \Big/ \frac{1}{n} = \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+1/n^2}} \xrightarrow[n\to\infty]{} 1$$

e $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(h) Converge pelo teste de comparação no limite, pois

$$\frac{n+1}{n^3+n} \Big/ \frac{1}{n^2} = \frac{n^3+n^2}{n^3+n} = \frac{1+1/n}{1+1/n^2} \xrightarrow[n\to\infty]{} 1$$

e $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

(i) Diverge pois

$$\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1} \geq \sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(j) Converge pois

$$\sum_{n=1}^{\infty} \left| \frac{2+\operatorname{sen} n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{3}{n^2},$$ que converge.

(k) Converge pois

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n} \leq \sum_{n=1}^{\infty} 4e^{-n} = \frac{4}{e} \cdot \frac{1}{1-\frac{1}{e}}$$

$$= 4/(e-1)$$

(l) $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge pelo teste da razão:

$$\frac{1}{(n+1)!} \Big/ \frac{1}{n!} = \frac{n!}{(n+1)\,n!} = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$$

(m) Diverge pelo teste de comparação no limite, pois

$$\frac{\operatorname{sen}\frac{1}{n}}{\frac{1}{n}} \xrightarrow[n\to\infty]{} 1 \quad \text{e} \quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(n) Converge pelo teste de comparação no limite, pois

$$\frac{\frac{1}{n}\tan\frac{1}{n}}{\frac{1}{n^2}} = \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \frac{\operatorname{sen}\frac{1}{n}}{\frac{1}{n}} \cdot \frac{1}{\cos\frac{1}{n}}$$

$$\xrightarrow[n\to\infty]{} 1$$

e $\sum_{n}^{\infty} \frac{1}{n^2}$ converge.

(o) Converge pelo teste de comparação no limite, pois

$$\frac{1}{\sqrt[3]{3n^4+1}} \Big/ \frac{1}{n^{4/3}} = \frac{1}{\sqrt[3]{3+1/n^4}} \xrightarrow[n\to\infty]{} \frac{1}{\sqrt[3]{3}}$$

e $\sum_{n}^{\infty} \frac{1}{n^{4/3}}$ converge.

(p) Converge pelo teste de comparação no limite, pois

$$\frac{1}{n\sqrt{n^2-1}} \Big/ \frac{1}{n^2} = \frac{1}{\sqrt{1-1/n^2}} \xrightarrow[n\to\infty]{} 1$$

e $\sum_{n}^{\infty} \frac{1}{n^2}$ converge.

(q) Diverge pelo teste de comparação no limite, pois

$$\frac{n^2+\cos n}{n^3} \Big/ \frac{1}{n} = \frac{1+(\cos n)/n^3}{1} \xrightarrow[n\to\infty]{} 1$$

e $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(r) Diverge pois

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(s) Diverge pelo teste de comparação no limite, pois

$$\frac{1}{n^{1+1/n}} \Big/ \frac{1}{n} = \frac{n}{n\sqrt[n]{n}} \xrightarrow[n\to\infty]{} 1$$

e $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Exercício 10. A representação decimal de um número entre 0 e 1 é $0, d_1 d_2 d_3 ...$, onde $d_i \in \{0, 1, 2, 3, .., 9\}$, que significa

$$0, d_1 d_2 d_3 ... = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + ...$$

Mostre que essa série sempre converge para qualquer escolha de $d_1, d_2, ... \in \{0, 1, 2, 3, ..., 9\}$.

Temos

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}}$$

$$= \frac{9}{10} \cdot \frac{10}{9} = 1 ,$$

Logo a série converge.

Exercício 11. Suponha que $\sum a_n$ e $\sum b_n$ sejam séries com termos positivos tais que

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

Mostre que se $\sum b_n$ converge, então $\sum a_n$ também converge.

Como $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$, existe $N_0 \in \mathbb{N}$ tal que se $n \geq N_0$, então

$$0 \leq \frac{a_n}{b_n} < 1$$

$$\Rightarrow a_n < b_n \quad (n \geq N_0)$$

Daí,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N_0-1} a_n + \sum_{n=N_0}^{\infty} a_n$$

$$\leq \sum_{n=1}^{N_0-1} a_n + \sum_{n=N_0}^{\infty} b_n < \infty$$

Logo,

$\sum a_n$ converge.

Exercício 12. Suponha que $\sum a_n$ e $\sum b_n$ sejam séries com termos positivos tais que

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

Mostre que se $\sum b_n$ diverge, então $\sum a_n$ também diverge.

Como $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, existe $N_0 \in \mathbb{N}$ tal que se $n \geqslant N_0$, então

$$\frac{a_n}{b_n} > 1$$

$$\Rightarrow a_n > b_n \quad (n \geqslant N_0)$$

Daí,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N_0-1} a_n + \sum_{n=N_0}^{\infty} a_n$$

$$\geqslant \sum_{n=N_0}^{\infty} b_n = \infty$$

Logo,

$\sum a_n$ diverge.

Exercício 13. Mostre que se $a_n \geqslant 0$ e $\sum a_n$ converge, então $\sum a_n^2$ também converge.

Como $\sum a_n$ converge,

$$\lim_{n \to \infty} a_n = 0$$

Daí, $\exists\ N_0 \in \mathbb{N}$ tal que se $n \geqslant N_0$ então

$$0 \leq a_n < 1$$

$$\Rightarrow 0 \leq a_n^2 < a_n \quad (n \geqslant N_0)$$

Logo,

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N_0-1} a_n^2 + \sum_{n=N_0}^{\infty} a_n^2$$

$$\leq \sum_{n=1}^{N_0-1} a_n^2 + \sum_{n=N_0}^{\infty} a_n < \infty$$

Assim,

$\sum a_n^2$ converge.

Exercício 14. Verifique se a série converge ou diverge:

(a) $\sum_{n=1}^{\infty}(-1)^n e^{-n}$ (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{2n+1}$ (c) $\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{\log n}$

(d) $\sum_{n=0}^{\infty}\frac{\operatorname{sen}\left(n+\frac{1}{2}\right)}{1+\sqrt{n}}$ (e) $\sum_{n=1}^{\infty}(-1)^n \operatorname{sen}\frac{\pi}{n}$ (f) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}$

(g) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2n+3}$ (h) $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

(a) Converge pois converge absolutamente:

$$\sum_{n=1}^{\infty}\frac{1}{e^n}=\frac{1}{e}\,\frac{1}{1-\frac{1}{e}}=\frac{1}{e-1}$$

(b) Converge pelo teste de Leibniz:

a série é alternada, $\frac{2}{2n+1}$ decresce

e $\lim_{n\to\infty}\frac{2}{2n+1}=0$

(c) Converge pelo teste de Leibniz:

a série é alternada, $\frac{1}{\log n}$ decresce

e $\lim_{n\to\infty}\frac{1}{\log n}=0$

(d) Converge pelo teste de Leibniz:

a série é alternada pois

$$\operatorname{sen}\left(n+\frac{1}{2}\right)\pi = (-1)^n$$

Além disso, $\dfrac{1}{1+\sqrt{n}}$ decresce e

e $\displaystyle\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$.

(e) Converge pelo teste de Leibniz:

a série é alternada, $\operatorname{sen}\frac{\pi}{n}$ decresce

e $\displaystyle\lim_{n\to\infty} \operatorname{sen}\frac{\pi}{n} = 0$.

(f) Converge pelo teste de Leibniz:

a série é alternada, $\dfrac{1}{\sqrt{n+1}}$ decresce

e $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

(g) Diverge pelo teste de comparação no limite, pois

$$\frac{\sqrt{n}}{2n+3} \Big/ \frac{1}{\sqrt{n}} = \frac{n}{2n+3} = \frac{1}{2+3/n} \xrightarrow{n\to\infty} \frac{1}{2}$$

e $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

(h) Converge pelo teste de Leibniz: a série é alternada e, além disso,

$$\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \frac{1}{(\sqrt{n+1}+\sqrt{n})},$$

que é decrescente e tende a zero quando $n \to \infty$.

Exercício 15. O que significa dizer que uma série é absolutamente convergente? E condicionalmente convergente?

$\sum a_n$ é absolutamente convergente se $\sum |a_n|$ converge. Ela é condicionalmente convergente se $\sum a_n$ converge mas $\sum |a_n|$ diverge.

Exercício 16. Determine se a série é absolutamente convergente, condicionalmente convergente ou divergente.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ (b) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$ (c) $\sum_{n=1}^{\infty} \frac{-n}{n^2+1}$

(d) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ (e) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ (f) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$

(a) Absolutamente convergente pois

$\sum \frac{1}{n^4}$ converge

(b) Divergente pois

$$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

(c) Divergente pois

$$\sum_{n=1}^{\infty} \frac{-n}{n^2+1} = -\sum_{n=1}^{\infty} \frac{n}{n^2+1}, \quad e$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} > \sum_{n=1}^{\infty} \frac{n}{2n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(d) Condicionalmente convergente.

De fato, $\frac{n}{\sqrt{n^3+2}} = \frac{1}{\sqrt{n+2/n^2}} \xrightarrow[n\to\infty]{} 0$.

Além disso,

$\dfrac{n}{\sqrt{n^3+2}}$ é decrescente pois

$\dfrac{n^2}{n^3+2}$ é decrescente. Com efeito,

Se $f(x) = \dfrac{x^2}{x^3+2}$,

$$f'(x) = \frac{2x(x^3+2) - 3x^2 \cdot x^2}{(x^3+2)^2}$$

$$= \frac{2x^4 + 4x - 3x^4}{(x^3+2)^2} = \frac{x(4-x^3)}{(x^3+2)^2}$$

$$< 0 \quad \text{se} \quad x > \sqrt[3]{4}$$

Logo, $\dfrac{n^2}{n^3+2}$ e, portanto, $\dfrac{n}{\sqrt{n^3+2}}$

é decrescente se $n \geq 2$.

Assim, o critério de Leibniz prova que a série converge.

Porém, não ocorre convergência absoluta por causa do teste da comparação no limite:

$$\frac{n}{\sqrt{n^3+2}} \Big/ \frac{1}{\sqrt{n}} = \frac{n^{3/2}}{\sqrt{n^3+2}} = \frac{1}{\sqrt{1+\frac{2}{n^3}}} \xrightarrow[n\to\infty]{} 1$$

e
$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty.$$

(e) Condicionalmente convergente:

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$ converge pelo critério de Leibniz para séries alternadas, mas $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverge pelo teste de comparação com integral, já que

$$\int_2^{\infty} \frac{dx}{x \log x} = \log(\log x)\Big|_2^{\infty} = \infty.$$

(f) Condicionalmente convergente:

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ converge pelo critério de Leibniz para séries alternadas, mas $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverge pois

$$\sum_{n=2}^{\infty} \frac{1}{\log n} \geq \sum_{n=2}^{\infty} \frac{1}{n} = \infty,$$

já que $f(x) = x - \log x$ satisfaz

$$f'(x) = 1 - \frac{1}{x} \geq 0 \quad \text{se } x \geq 1$$

$$\Rightarrow f(x) \geq f(1) = 1 > 0$$

$$\Rightarrow x > \log x \quad (x \geq 1)$$

Exercício 17. Para quais valores de p a série é convergente?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$ (c) $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1}\frac{(\log n)^p}{n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

(a) Converge para $p>0$ pelo critério de Leibniz.

(b) Converge para $\forall p \in \mathbb{R}$ pelo critério de Leibniz.

(c) Converge para $\forall p \in \mathbb{R}$ pelo critério de Leibniz, pois

$$\lim_{n\to\infty} \frac{(\log n)^p}{n} = 0 \quad \forall p \in \mathbb{R}$$

e

$$f(x) = \frac{(\log x)^p}{x} \quad \text{é decrescente}$$

para x suficientemente grande, já que

$$f'(x) = \frac{p(\log x)^{p-1} - (\log x)^p}{x^2}$$

$$= \frac{(\log x)^{p-1}}{x^2}\cdot(p - \log x) < 0$$

se $x > e^p$.

(e) Converge para $p > 0$ pelo critério de Leibniz.

Exercício 18. Em um dos exercícios anteriores, vimos que a constante de Euler-Mascheroni γ satisfaz

$$\gamma = \lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} - \log n$$

Seja h_n a soma parcial da série harmônica e a_n a soma parcial da série harmônica alternada.

(a) Mostre que (a_n) converge.

(b) Mostre que $a_{2n} = h_{2n} - h_n$

(c) Use os itens anteriores e a caracterização de γ para provar que

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \log 2$$

(a) $\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge pelo critério de Leibniz.

(b) $h_{2n} - h_n = h_{2n} - \frac{2 \cdot h_n}{2} =$

$$= \sum_{k=1}^{n} \frac{1}{2k-1} + \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{2}{2k} \right)$$

$$= \sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k}$$

$$= \sum_{k=1}^{2n} (-1)^{n-1} \cdot \frac{1}{n} = a_{2n}$$

(c) Por (b),

$h_{2n} - h_n = a_{2n}$ converge, e temos

$$\lim_{n\to\infty} h_{2n} - h_n = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_n$$

Por outro lado,

$$t_n = h_n - \log n \xrightarrow[n\to\infty]{} \gamma$$

Note que

$$h_n = t_n + \log n$$

$$\Rightarrow h_{2n} - h_n = t_{2n} - t_n + \log 2n - \log n$$

$$= t_{2n} - t_n + \log 2 + \cancel{\log n} - \cancel{\log n}$$

$$\Rightarrow a_{2n} = t_{2n} - t_n + \log 2$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{2n} = \gamma - \gamma + \log 2$$

$$= \log 2.$$

Exercício 19. Use o teste da razão para determinar se a série converge:

(a) $\sum_{n=1}^{\infty} \frac{n}{5^n}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2^n n^3}$ (c) $\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$ (d) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(e) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ (f) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (g) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

(a) Converge pois

$$\frac{n+1}{5^{n+1}} \frac{5^n}{n} = \frac{n+1}{n} \cdot \frac{1}{5} \xrightarrow[n\to\infty]{} \frac{1}{5} < 1$$

(b) Diverge pois

$$\frac{3^{n+1} \cdot n^3 \cdot 2^n}{2^{n+1}(n+1)^3 \cdot 3^n} = \frac{3}{2} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^3} \xrightarrow[n\to\infty]{} \frac{3}{2} > 1.$$

(c) Diverge pois

$$\frac{(n+1)\,\pi^{n+1} \cdot 3^{n-1}}{3^n \cdot n \cdot \pi^n} = \frac{n+1}{n} \cdot \frac{\pi}{3} \xrightarrow[n\to\infty]{} \frac{\pi}{3} > 1$$

(d) Diverge pois

$$\frac{(n+1)!\ 100^n}{100^{n+1}\ n!} = \frac{n}{100} \xrightarrow[n\to\infty]{} \infty$$

(e) Converge pois

$$\frac{3^{n+1}(2n+1)!}{(2n+3)!\ 3^n} = \frac{3}{(2n+3)(2n+2)} \xrightarrow[n\to\infty]{} 0 < 1$$

(f) Converge pois

$$\frac{(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot n!} = \frac{(n+1)}{(n+1)\cdot\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$$

(g) Diverge pois

$$\frac{(2n+2)!\,[n!]^2}{[(n+1)!]^2\,(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \frac{(2+2/n)(2+1/n)}{(1+1/n)^2} \xrightarrow[n\to\infty]{} 4 > 1$$

Exercício 20. Use o teste da raiz para determinar se a série converge:

(a) $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ (b) $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(\log n)^n}$ (c) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ (d) $\sum_{n=0}^{\infty}(\arctan n)^n$

(a) converge pois

$$\frac{n^2+1}{2n^2+1} = \frac{1+1/n^2}{2+1/n^2} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

(b) Converge pois

$$\frac{1}{\log n} \xrightarrow[n\to\infty]{} 0 < 1$$

(c) Diverge pois

$$\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e > 1$$

(d) Diverge pois

$$\arctan n \xrightarrow[n\to\infty]{} \frac{\pi}{2} > 1$$

Exercício 21. Determine se série é absolutamene convergente, condicionalmente convergente ou divergente:

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n \log n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{(-9)^n}{n10^{n+1}}$ (c) $\sum_{n=2}^{\infty} \left(\frac{n}{\log n}\right)^n$ (d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\log n}$

(e) $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$ (f) $\sum_{n=1}^{\infty} \frac{\text{sen}(n\pi/6)}{1+n\sqrt{n}}$ (g) $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

(a) Converge condicionalmente pelo critério de Leibniz, pois a série é alternada, $\lim_{n\to\infty} \frac{\log n}{n} = 0$ e

$f(x) = \frac{\log x}{x}$ é decrescente pois

$f'(x) = \frac{1-\log x}{x^2} < 0$ se $x \geq e$.

Por outro lado a série diverge absolutamente pois

$$\sum_{n=1}^{\infty} \frac{\log n}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(b) Converge absolutamente pois

$$\sum_{n=1}^{\infty} \frac{9^n}{n\cdot 10^{n+1}} \leq \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{1-\frac{9}{10}} = 10$$

(c) Diverge pois

$$\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^n = \infty$$

(d) Converge condicionalmente pelo critério de Leibniz. Por outro lado, a série diverge absolutamente pois

$$f(x) = \sqrt{x} - \log x > 0 \quad (x \geqslant 4)$$

pois

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x} = \frac{\sqrt{x}-2}{2x} > 0$$

se $x \geqslant 4$

$$\Rightarrow f(x) \geqslant f(4) = 2 - \log 4 > 0$$

$$= 2(1 - \log 2) > 0$$

Logo,

$$\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}\log n} \geqslant \sum_{n=4}^{\infty} \frac{1}{(\sqrt{n})^2} = \infty$$

(e) Diverge pois

$$\lim_{n\to\infty} \frac{n\,5^{2n}}{10^{n+1}} = \lim_{n\to\infty} \frac{n \cdot 10^{n}}{10 \cdot 10^{n}} = \infty$$

(f) Converge absolutamente pois

$$\sum_{n=1}^{\infty} \frac{|\operatorname{sen}(n\pi/6)|}{1+n\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{2n^{3/2}},$$ que converge.

(g) Converge absolutamente pelo teste da raiz, pois

$$\sqrt[n]{2} - 1 \xrightarrow[n\to\infty]{} 0$$

Exercício 22. Defina $a_1 = 1$ e

$$a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$$

Determine se $\sum a_n$ converge ou diverge.

Converge pelo teste da razão:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2+\cos n}{\sqrt{n}}\right| \leq \frac{3}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

Exercício 23. Para quais valores de $k \in \mathbb{N}$ a série abaixo converge?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

Teste da razão:

$$\frac{[(n+1)!]^2 \, (kn)!}{(kn+k)! \, (n!)^2} = \frac{(n+1)^2}{(kn+k)\cdots(kn+1)} = \alpha_n(k)$$

Se $k=1$, $\alpha_n(1) = n+1 \xrightarrow[n\to\infty]{} \infty$

Se $k=2$, converge pois

$$\alpha_n(2) = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1+1/n}{4\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}$$

$$\xrightarrow[n\to\infty]{} \frac{1}{4} < 1$$

Para $k>2$,

$$\alpha_n(k) \leq \frac{(n+1)^2}{(n+1)^k} = (n+1)^{2-k}$$

$$\xrightarrow[n\to\infty]{} 0$$

Logo a série converge para $k \geqslant 2$.

Exercício 1. Encontre o raio de convergência e o intervalo de convergência das séries de potências abaixo:

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty} \sqrt{n}x^n$ (c) $\sum_{n=1}^{\infty} \frac{n}{5^n}x^n$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ (e) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(f) $\sum_{n=1}^{\infty} n^n x^n$ (g) $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \log n}$ (h) $\sum_{n=4}^{\infty} \frac{\log n}{n}x^n$ (i) $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\log n)^2}$

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

Teste da raiz:

$$\frac{|x|}{\sqrt[n]{n}} \xrightarrow[n\to\infty]{} |x| < 1 \quad \text{se} \quad |x| < 1$$

$\Rightarrow$ Raio de convergência $R = 1$

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge (série harmônica)

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge (teste de Leibniz)

$\Rightarrow$ Intervalo de convergência

$$I = [-1, 1)$$

(6) $\sum_{n=1}^{\infty} \sqrt{n}\, x^n$

Teste da raiz:

$$\left(\sqrt[n]{n}\right)^{1/2} |x| \xrightarrow[n\to\infty]{} |x| < 1 \quad \text{se} \quad |x| < 1$$

$\Rightarrow$ Raio de convergência $R = 1$

Se $|x| = 1$,

$$\sqrt{n}\,|x|^n = \sqrt{n} \xrightarrow[n\to\infty]{} \infty$$

Logo, como o termo geral não tende a zero, a série diverge se $|x| = 1$

$\Rightarrow$ Intervalo de convergência

$$I = (-1, 1)$$

(c) $\sum_{n=1}^{\infty} \frac{n}{5^n} x^n$

Teste da raiz:

$$\left(\sqrt[n]{n}\right) \frac{|x|}{5} \xrightarrow[n\to\infty]{} \frac{|x|}{5} < 1 \quad \text{se} \quad |x| < 5$$

$\Rightarrow$ Raio de convergência $R = 5$

Se $|x| = 5$,

$$n \frac{|x|^n}{5^n} = n \xrightarrow[n\to\infty]{} \infty$$

Logo, como o termo geral não tende a zero, a série diverge se $|x| = 5$

$\Rightarrow$ Intervalo de convergência

$$I = (-5, 5)$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

Teste da raiz:

$$\frac{|x|}{\left(\sqrt[n]{n}\right)^{1/3}} \xrightarrow[n\to\infty]{} |x| < 1 \quad \text{se} \quad |x| < 1$$

$\Rightarrow$ Raio de convergência $R = 1$

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverge $\left(\sum \frac{1}{n^\alpha} \text{ diverge se } \alpha < 1\right)$

$x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converge (teste de Leibniz)

$\Rightarrow$ Intervalo de convergência

$$I = (-1, 1]$$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Teste da razão:

$$\frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0$$

para todo $x \in \mathbb{R}$

$\Rightarrow$ Raio de convergência $R = \infty$

$\Rightarrow$ Intervalo de convergência

$I = (-\infty, \infty)$

(f) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

Teste da raiz:

$$n\,|x| \xrightarrow[n\to\infty]{} \begin{cases} \infty & \text{se} \quad |x| \neq 0 \\ 0 & \text{se} \quad |x| = 0 \end{cases}$$

$\Rightarrow$ Raio de convergência $R = 0$

$\Rightarrow$ Intervalo de convergência $I = \{0\}$

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \log n}$

Teste da raiz:

$$\frac{|x+2|}{2\sqrt[n]{\log n}} \xrightarrow[n\to\infty]{} \frac{|x+2|}{2} < 1 \quad \text{se} \quad |x+2| < 2,$$

pois $g(y) = y - \log y > 0$ se $y \geq 1$,

já que $g'(y) = 1 - \dfrac{1}{y} = \dfrac{y-1}{y} \geq 0$ se $y \geq 1$

e $g(1) = 1 > 0$.

Daí,

$$1 \leq \sqrt[n]{\log n} \leq \sqrt[n]{n} \qquad (n \geq 3)$$

$$\Rightarrow 1 \leq \lim_{n\to\infty} \sqrt[n]{\log n} \leq 1$$

$\Rightarrow$ Raio de convergência $R = 1$

$x = 0$: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverge pois

$$\sum_{n=2}^{\infty} \frac{1}{\log n} \geq \sum_{n=2}^{\infty} \frac{1}{n} = \infty$$

$x = -4$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}$ converge (teste de Leibniz)

$\Rightarrow$ Intervalo de convergência

$$I = [-4, 0)$$

(h) $\displaystyle\sum_{n=4}^{\infty} \frac{\log n}{n} x^n$

Teste da raiz:

$$\frac{\sqrt[n]{\log n}}{\sqrt[n]{n}} |x| \xrightarrow[n \to \infty]{} |x| < 1 \quad \text{se} \quad |x| < 1,$$

pois vimos no item anterior que

$$\lim_{n\to\infty} \sqrt[n]{\log n} = 1$$

$\Rightarrow$ Raio de convergência $R = 1$

$$x = 1: \sum_{n=4}^{\infty} \frac{\log n}{n} \geq \sum_{n=4}^{\infty} \frac{1}{n} = \infty$$

$x = -1$: $\displaystyle\sum_{n=4}^{\infty} (-1)^n \frac{\log n}{n}$ converge pelo teste de Leibniz, pois se $g(y) = \dfrac{\log y}{y}$,

$$g'(y) = \frac{1 - \log y}{y} < 0 \text{ se } y \geq 4,$$

logo $\dfrac{\log n}{n}$ é decrescente se $n \geq 4$.

Além disso,

$$\lim_{n\to\infty} \frac{\log n}{n} = 0.$$

$\Rightarrow$ Intervalo de convergência $I = [-1, 1)$

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\log n)^2}$

Teste da raiz:

$$\frac{|x|^2}{\sqrt[n]{n}\,\left(\sqrt[n]{\log n}\right)^2} \xrightarrow[n\to\infty]{} |x|^2 < 1 \quad \text{se} \quad |x| < 1,$$

pois vimos no item (g) que

$$\lim_{n\to\infty} \sqrt[n]{\log n} = 1$$

$\Rightarrow$ Raio de convergência $R = 1$

Se $|x| = 1$,

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converge pelo teste

da integral, pois $\dfrac{1}{x(\log x)^2}$ é uma

função decrescente e

$$\int_2^\infty \frac{dx}{x(\log x)^2} = \int_2^\infty \frac{d}{dx}\left(\frac{-1}{\log x}\right) dx = \left.\frac{-1}{\log x}\right|_2^\infty$$

$$= \frac{1}{\log 2}.$$

$\Rightarrow$ Intervalo de convergencia $I = [-1, 1]$

Exercício 2. Se $k \in \mathbb{N}$, encontre o raio de convergência da série

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

Teste da razão:

$$\frac{[(n+1)!]^k \cdot (kn)! \cdot |x|^{n+1}}{[k(n+1)]! \cdot (n!)^k \quad |x|^n}$$

$$= \frac{(n+1)^k \cdot |x|}{(kn+k)\cdot(kn+k-1)\cdots(kn+1)}$$

termos de grau < k

$$= \frac{(n^k + kn^{k-1} + \cdots)}{kn^k + \cdots} \cdot |x|$$

$$= \frac{1 + k/n + \cdots}{k + \cdots} \cdot |x| \xrightarrow[n\to\infty]{} \frac{|x|}{k} < 1$$

se $|x| < k$.

$\Rightarrow$ Raio de convergência $R = k$.

Exercício 3. Sejam p, q números reais, $p < q$. Encontre uma série de potências cujo intervalo de convergência seja:

(a) (p, q) (b) $(p, q]$ (c) $[p, q)$ (d) $[p, q]$

(a)
$$\sum_{n=0}^{\infty} n \left(\frac{2}{q-p}\right)^n \left(x - \frac{p+q}{2}\right)^n$$

Teste da raiz:

$$\sqrt[n]{n} \left| \frac{x - \left(\frac{p+q}{2}\right)}{\frac{q-p}{2}} \right| \xrightarrow[n\to\infty]{} \left| \frac{x - \left(\frac{p+q}{2}\right)}{\frac{q-p}{2}} \right| < 1$$

se $\left| x - \frac{p+q}{2} \right| < \frac{q-p}{2}$

$$\iff \frac{p-q}{2} + \frac{p+q}{2} < x < \frac{p+q}{2} + \frac{q-p}{2}$$

$$\iff \quad p < x < q.$$

Se $\left| x - \frac{p+q}{2} \right| = \frac{q-p}{2}$,

$$\frac{n\left| x - \frac{p+q}{2} \right|}{\frac{q-p}{2}} = n \xrightarrow[n\to\infty]{} \infty,$$

então a série diverge se

$$\left|x - \frac{p+q}{2}\right| = \frac{q-p}{2}$$

$\Rightarrow$ Intervalo de convergência $I = (p, q)$

(6) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{2}{q-p}\right)^n \left(x - \frac{p+q}{2}\right)^n$$

Teste da raiz:

$$\frac{1}{\sqrt[n]{n}} \cdot \left(\frac{2}{q-p}\right) \cdot \left|x - \frac{p+q}{2}\right| \xrightarrow[n\to\infty]{} \left(\frac{2}{q-p}\right) \cdot \left|x - \frac{p+q}{2}\right|$$

< 1 se

$$\left|x - \frac{p+q}{2}\right| < \frac{q-p}{2}$$

Se $x = p$,

$$\sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{-2}{q-p}\right)^n \left[-\left(\frac{q-p}{2}\right)\right]^n = \sum_{n=0}^{\infty} \frac{1}{n} = \infty$$

Se $x = q$,

$$\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{-2}{q-p}\right)^n\left(\frac{q-p}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}, \quad \text{que}$$

converge pelo teste de Leibniz.

$\Rightarrow$ Intervalo de convergência $I = [p, q]$

(c) $$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2}{q-p}\right)^n\left(x - \frac{p+q}{2}\right)^n$$

Teste da raiz:

$$\frac{1}{\sqrt[n]{n}} \cdot \left(\frac{2}{q-p}\right) \cdot \left|x - \frac{p+q}{2}\right| \xrightarrow[n\to\infty]{} \left(\frac{2}{q-p}\right) \cdot \left|x - \frac{p+q}{2}\right|$$

< 1 se

$$\left|x - \frac{p+q}{2}\right| < \frac{q-p}{2}$$

Se $x = q$,

$$\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{2}{q-p}\right)^n\left(\frac{q-p}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n} = \infty$$

Se $x = p$,

$$\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{2}{q-p}\right)^n \left[-\left(\frac{q-p}{2}\right)\right]^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n},$$ que

converge pelo teste de Leibniz.

$\Rightarrow$ Intervalo de convergência $I = [p, q)$

(d) $$\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{2}{q-p}\right)^n \left(x - \frac{p+q}{2}\right)^n$$

Teste da raiz:

$$\frac{1}{(\sqrt[n]{n})^2}\left(\frac{2}{q-p}\right) \cdot \left|x - \frac{p+q}{2}\right| \xrightarrow[n\to\infty]{} \left(\frac{2}{q-p}\right) \cdot \left|x - \frac{p+q}{2}\right|$$

< 1 se

$$\left|x - \frac{p+q}{2}\right| < \frac{q-p}{2}$$

Se $\left| x - \frac{p+q}{2} \right| = \frac{q-p}{2}$,

$$\sum_{n=0}^{\infty} \frac{1}{n^2} \left(\frac{2}{q-p}\right)^n \left(\frac{q-p}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n^2}$$, que

converge (por ex, pelo teste da integral).

$\Rightarrow$ Intervalo de convergência $I = [p, q]$

Exercício 4. É possível encontrar uma série de potências com intervalo de convergência $[0, \infty)$? Explique.

Não. Considere a série

$$\sum_{n=0}^{\infty} a_n (x-\alpha)^n,$$

onde $\alpha \in \mathbb{R}$.

Vimos que se

$$\sum_{n=0}^{\infty} a_n (x_0-\alpha)^n$$

converge, então

$$\sum_{n=0}^{\infty} a_n (x-\alpha)^n$$

converge se $|x-\alpha| < |x_0-\alpha|$.

Logo, uma série que converge em $[0, \infty)$ implica que, tomando

$$x_0 = \alpha + K > 0,$$

$$\sum_{n=0}^{\infty} a_n K^n \quad \text{converge,}$$

e então

$$\sum_{n=0}^{\infty} a_n (x-\alpha)^n$$

converge para todo x tal que

$$\alpha - K < x < \alpha + K$$

Fazendo $K \to \infty$, vemos que a série converge $\forall\, x \in \mathbb{R}$, sendo impossível que seu intervalo de convergência seja apenas $[0, \infty)$.

Exercício 5. Encontre uma representação em série de potências para cada uma das funções abaixo e determine o seu raio de convergência:

(a) $f(x) = \dfrac{1}{1+x}$ (b) $f(x) = \dfrac{1}{1-x^2}$ (c) $f(x) = \dfrac{2}{3-x}$

(d) $f(x) = \dfrac{x^2}{x^4+16}$ (e) $f(x) = \dfrac{x+a}{x^2+a^2}$, $(a > 0)$ (f) $f(x) = \dfrac{2x-4}{x^2-4x+3}$

(g) $f(x) = \dfrac{2x+3}{x^2+3x+2}$

(a) $$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow R = 1$$

(b) $$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \Rightarrow R = 1$$

(c) $$\frac{2}{3-x} = \frac{2}{3} \cdot \frac{1}{1 - x/3} = \sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n$$

Teste da raiz:

$$\sqrt[n]{\frac{2}{3}}\, \frac{|x|}{3} \xrightarrow[n\to\infty]{} \frac{|x|}{3} < 1 \text{ se } |x| < 3$$
$$\Rightarrow R = 3$$

(d) $$\frac{x^2}{x^4+16} = \frac{x^2}{16} \cdot \frac{1}{1+(x/2)^4}$$

$$= \frac{x^2}{16} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}} \cdot x^{4n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4(n+1)}} \cdot x^{4n}$$

Teste da raiz:

$$\frac{1}{\sqrt[n]{16}\cdot 16}|x|^4 \xrightarrow[n\to\infty]{} \frac{|x|^4}{16} < 1 \quad \text{se} \quad |x| < 2$$

$$\Rightarrow R = 2$$

(e) $$\frac{x+a}{x^2+a^2} = \frac{x}{a^2}\cdot\frac{1}{1+(x/a)^2} + \frac{1}{a}\cdot\frac{1}{1+(x/a)^2}$$

$$= \frac{x}{a^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{a^{2n}}x^{2n} + \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{a^{2n}}x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{a\cdot a^{2n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{a\cdot a^{2n}}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor n/2\rfloor}}{a^{n+1}}\cdot x^n, \quad \text{onde}$$

$\lfloor n/2 \rfloor \in \mathbb{Z}$ satisfaz

$$\lfloor n/2\rfloor \leq \frac{n}{2} < \lfloor n/2\rfloor + 1$$

Teste da raiz:

$$\frac{1}{\sqrt[n]{a}} \cdot \frac{|x|}{a} \xrightarrow[n\to\infty]{} \frac{|x|}{a} < 1 \quad \text{se} \quad |x| < a$$

$$\Rightarrow R = a$$

(f) $$\frac{2x-4}{x^2-4x+3} = \frac{2x-4}{(x-1)(x-3)}$$

$$= \frac{a}{x-1} + \frac{b}{x-3} = \frac{(a+b)x - 3a - b}{(x-1)(x-3)}$$

$$\Rightarrow \begin{cases} a+b = 2 \Rightarrow b = 2-a \\ 3a+b = 4 \Rightarrow 3a+2-a = 4 \Rightarrow a = 1 \\ \qquad\qquad\qquad\qquad \Rightarrow b = 1 \end{cases}$$

$$\Rightarrow f(x) = \frac{-1}{1-x} - \frac{1}{3} \cdot \frac{1}{1-x/3}$$

$$= -\sum_{n=0}^{\infty} x^n - \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} \cdot x^n$$

$$= \sum_{n=0}^{\infty} -\left(1 + \frac{1}{3^n}\right) x^n = \sum_{n=0}^{\infty} -\left(\frac{3^n+1}{3^n}\right) x^n$$

Teste da raiz:

Devemos avaliar $\dfrac{\sqrt[n]{3^n+1}}{3} |x|$. Note que

$$|x| = \frac{\sqrt[n]{3^n+0}}{3}|x| < \frac{\sqrt[n]{3^n+1}}{3}|x|$$

$$< \frac{\sqrt[n]{3^n+3^n}}{3}|x| = \sqrt[n]{2}\,|x|$$

$$\Rightarrow \quad |x| \leq \lim_{n\to\infty} \frac{\sqrt[n]{3^n+1}}{3}|x| \leq |x|$$

Assim, $R = 1$.

(g) $$\frac{2x+3}{x^2+3x+2} = \frac{2x+3}{(x+1)(x+2)}$$

$$= \frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{\frac{x}{2}+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(1+\frac{1}{2^{n+1}}\right)x^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2^{n+1}+1}{2^{n+1}}\right)x^n$$

Semelhantemente ao item anterior, vemos pelo teste da raiz que $R = 1$.

Exercício 6. Use derivação para encontrar uma representação em série de potências para

$$f(x) = \frac{1}{(1+x)^2}$$

Determine seu raio de convergência.

$$\frac{1}{(1+x)^2} = \frac{d}{dx}\frac{-1}{1+x} = \frac{d}{dx} - \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n\, x^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

Teste da raiz:

$$\sqrt[n]{n+1}\,|x| \xrightarrow[n\to\infty]{} |x| < 1 \quad \text{se} \quad |x| < 1,$$

pois

$$\sqrt[n]{n} \leq \sqrt[n]{n+1} \leq \sqrt[n]{2n} = \sqrt[n]{2} \cdot \sqrt[n]{n}$$

$$\Rightarrow 1 \leq \lim_{n\to\infty} \sqrt[n]{n+1} \leq 1$$

Logo, o raio de convergência é

$$R = 1.$$

Exercício 7. Encontre uma representação em série de potências para

$$f(x) = \frac{1}{(1+x)^3}$$

e determine seu raio de convegência.

Temos, usando o exercício anterior,

$$\frac{1}{(1+x)^3} = \frac{d}{dx}\left(\frac{-1}{2}\right)\frac{1}{(1+x)^2}$$

$$= \frac{d}{dx}\frac{-1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{n(n+1)}{2}x^{n-1} = \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)}{2}x^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\binom{n+2}{2}x^n,$$

que possui mesmo raio de convergência de sua primitiva. Pelo exercício anterior, $R=1$.

Exercício 8. Encontre uma série de potências para $\log(1-x)$ utilizando integração. Em seguida, faça $x = 1/2$ e encontre uma expressão de $\log 2$ como uma série.

$$\log(1-x) = \int_1^{1-x} \frac{du}{u} = -\int_0^x \frac{dt}{1-t} \qquad \begin{cases} t = 1-u \\ dt = -du \end{cases}$$

$$= -\int_0^x \sum_{n=0}^{\infty} t^n \, dt = \sum_{n=0}^{\infty} -\frac{1}{n+1} t^{n+1} \Big|_0^x$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n} \cdot x^n$$

Fazendo $x = 1/2$:

$$\log \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{2^n}$$

$$\Rightarrow \log 2 = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{2^n}$$

Exercício 9. Represente cada uma das seguintes funções como série de potências:

(a) $f(x) = \dfrac{1+x}{(1-x)^2}$ (b) $f(x) = \log(5-x)$ (c) $f(x) = x^2 \arctan x^3$

(d) $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$

(a)
$$\frac{1+x}{(1-x)^2} = -\frac{(1-x)}{(1-x)^2} + 2\cdot\frac{1}{(1-x)^2}$$
$$= -\frac{1}{1-x} + 2\cdot\frac{1}{(1-x)^2}$$
$$= -\frac{1}{1-x} + 2\cdot\frac{d}{dx}\frac{1}{1-x}$$
$$= -\sum_{n=0}^{\infty} x^n + 2\cdot\sum_{n=1}^{\infty} n\,x^{n-1}$$
$$= -\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}(n+1)x^n$$
$$= \sum_{n=0}^{\infty}(2n+1)\,x^n$$

(b)
$$\log(5-x) = \log[5(1-x/5)]$$
$$= \log 5 + \log\left(1-\frac{x}{5}\right)$$

$$= \log 5 - 5 \int \frac{dx}{1 - \frac{x}{5}}$$

$$= \log 5 - 5 \sum_{n=0}^{\infty} \int \frac{x^n}{5^n} dx$$

$$= \log 5 - 5 \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n \cdot (n+1)}$$

$$= \log 5 - \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^{n-2}}$$

(c) Lembre que

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\Rightarrow x^2 \arctan x^3 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{6n+3}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{6n+5}}{2n+1}$$

(d) $f(x) = \log\left(\dfrac{1+x}{1-x}\right) = \log(1+x) + \log(1-x)$

Temos

$$\log(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\Rightarrow \log(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-1)^n x^n = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Logo,

$$f(x) = \log(1+x) - \log(1-x)$$

$$= \sum_{n=1}^{\infty} \frac{\left[1 + (-1)^{n+1}\right]}{n} x^n = \sum_{n=0}^{\infty} \frac{2 \cdot x^{2n+1}}{2n+1}$$

Exercício 10. Calcule as integrais indefinidas como séries de potências e determine os raios de convergência.

(a) $\int \frac{t}{1-t^8}\,dt$ (b) $\int x^2 \log(1+x^2)\,dx$ (c) $\int \frac{\arctan x}{x}\,dx$

(d) $\int x \log(1+x^2)\,dx$

(a) $$\int \frac{t}{1-t^8}dt = \int t \cdot \sum_{n=0}^{\infty} t^{8n}\,dt = \sum_{n=0}^{\infty} \int t^{8n+1}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$$

Teste da raiz : $\frac{\sqrt[n]{|t|^2} \cdot |t|^8}{\sqrt[n]{8n+2}} \to |t|^8 < 1$ se $|t|<1$

$\Rightarrow$ Raio de convergência $R = 1$.

(b) $$\int x^2 \log(1+x^2)\,dx = \int x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n}\,dx$$

$$= \sum_{n=1}^{\infty} \int \frac{(-1)^{n+1}}{n} x^{2n+2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{2n+3},$$

Teste da raiz : $\frac{\sqrt[n]{|x|^3} \cdot |x|^2}{\sqrt[n]{n+1}} \to |x|^2 < 1$ se $|x|<1$

$\Rightarrow$ Raio de convergência $R = 1$.

(c) Lembre que

$$\arctan x = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (|x| < 1)$$

$$\Rightarrow \int \frac{\arctan x}{x}\, dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{2n+1}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)\, x^{2n+1}}{(2n+1)^2} \qquad (|x| < 1)$$

O raio de convergência é o mesmo da série de $\arctan x$ $\Rightarrow$ $R = 1$.

(d)
$$\log(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \qquad (|x| < 1)$$

$$\Rightarrow \int x^2 \log(1+x^2)\, dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^{2n+2}\, dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+3}}{n(2n+3)}$$

O raio de convergência é o mesmo da série de $\log(1+x) \Rightarrow R=1$.

Exercício 11. Mostre que

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

é solução da equação diferencial

$$f'(x) = f(x).$$

Em seguida, prove que $e^x = f(x)$.

Temos

$$f'(x) = \sum_{n=0}^{\infty} \frac{n \cdot x^{n-1}}{n(n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

Para ver que $f(x) = e^x$, note que

$$[f(x)e^{-x}]' = f'(x)e^{-x} - f(x)e^{-x}$$

$$= (f'(x) - f(x))e^{-x} = 0$$

Logo, $f(x)e^{-x}$ é constante. Daí,

$$f(x)e^{-x} = f(0)e^{-0} = 1$$

$$\Rightarrow f(x) = e^x.$$

Isso vale para todo $x \in \mathbb{R}$, pois $f(x)$ tem raio de convergência $R = \infty$

pelo teste da razão:

$$\frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0 \quad \forall\, x \in \mathbb{R}$$

Exercício 12. Use a série de potências de $\arctan x$ para demonstrar a seguinte expressão de π como série infinita:

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

Temos que

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad (|x| < 1)$$

$$\Rightarrow \frac{\pi}{6} = \arctan \frac{1}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cdot \frac{1}{(\sqrt{3})^{2n} \cdot \sqrt{3}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n} \cdot \frac{1}{\sqrt{3}}$$

$$\Rightarrow \pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n}$$

Exercício 13. Determine o intervalo de convergência de

$$\sum_{n=1}^{\infty} n^3 x^n$$

e calcule o valor da sua soma.

Teste da raiz:

$$\sqrt[n]{n^3} \cdot \sqrt[n]{|x|^n} = \left(\sqrt[n]{n}\right)^3 \cdot |x| \xrightarrow[n\to\infty]{} |x|$$

Raio de convergência $R = 1$.

Como, para $|x| = 1$,

$$|n^3 x^n| = |n^3| \xrightarrow[n\to\infty]{} \infty,$$

a série diverge em $|x| = 1$.

Logo, o intervalo de convergência é $I = (-1, 1)$.

Vamos calcular o valor da soma.

Seja

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad (|x| < 1)$$

Temos

$$f'(x) = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$f''(x) = \sum_{n=0}^{\infty} n(n+1)\,x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2)\,x^{n}$$

$$= \sum_{n=0}^{\infty} (n^2 + 3n + 2)\,x^{n}$$

$$f'''(x) = \sum_{n=0}^{\infty} n(n+1)(n+2)\,x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)\,x^{n}$$

$$= \sum_{n=0}^{\infty} (n^2 + 3n + 2)(n+3)\,x^{n}$$

$$= \sum_{n=0}^{\infty} (n^3 + 3n^2 + 3n^2 + 9n + 2n + 6)\,x^{n}$$

$$= \sum_{n=0}^{\infty} (n^3 + 6n^2 + 11n + 6)\,x^{n}$$

Logo, como

$$n^3 = (n^3 + 6n^2 + 11n + 6)$$
$$- 6(n^2 + 3n + 2) + 7(n+1) - 1,$$

segue que

$$\sum_{n=0}^{\infty} n^3 x^n = \sum_{n=0}^{\infty} (n^3 + 6n^2 + 11n + 6)\, x^n$$
$$- 6\sum_{n=0}^{\infty} (n^2 + 3n + 2)\, x^n + 7\sum_{n=0}^{\infty} (n+1)\, x^n - \sum_{n=0}^{\infty} x^n$$
$$= f'''(x) - 6f''(x) + 7f'(x) - f(x).$$

Lembrando que

$$f(x) = \frac{1}{1-x} \Rightarrow \begin{cases} f'(x) = \dfrac{1}{(1-x)^2} \\ f''(x) = \dfrac{2}{(1-x)^3} \\ f'''(x) = \dfrac{6}{(1-x)^4}, \end{cases}$$

segue que

$$\sum_{n=0}^{\infty} n^3 x^n = \frac{6}{(1-x)^4} - \frac{12}{(1-x)^3} + \frac{7}{(1-x)^2} - \frac{1}{1-x}$$

Exercício 1. Encontre a série de Maclaurin das seguintes funções, bem como seu raio de convergência:

(a) $f(x) = (1-x)^{-2}$ (b) $f(x) = \cos x$ (c) $f(x) = 2^x$ (d) $f(x) = \operatorname{senh} x$
(e) $f(x) = \log(1+x)$

(a) Note que se

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (|x|<1),$$

temos

$$f(x) = g'(x) = \frac{1}{(1-x)^2}.$$

Em $|x|<1$, podemos derivar termo a termo:

$$f(x) = g'(x) = \sum_{n=0}^{\infty} n x^{n-1}$$

O raio de convergência é o mesmo da série de g: $(|x|<1)$.

(b)
$$\begin{cases} f(x) = \cos x \Rightarrow f(0) = 1 \\ f'(x) = -\operatorname{sen} x \Rightarrow f'(0) = 0 \\ f''(x) = -\cos x \Rightarrow f''(0) = -1 \\ f'''(x) = \operatorname{sen} x \Rightarrow f'''(0) = 0 \end{cases}$$

Comportamento cíclico: se $m \in \mathbb{N}^*$, $k \in \{0, 1, 2, 3\}$, então

$$f^{(4m+k)}(0) = \begin{cases} 1, & \text{se } k = 0 \\ 0, & \text{se } k = 1 \text{ ou } 3 \\ -1, & \text{se } k = 2 \end{cases}$$

Daí,
$$f^{(2n)}(0) = \begin{cases} 1 & \text{se } n \text{ é par} \\ -1 & \text{se } n \text{ é ímpar} \end{cases}$$

$$\Rightarrow f^{(2n)}(0) = (-1)^n.$$

A série de Taylor fica

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Raio de convergência:

- Teste da razão:

$$\frac{(2n)!\ |x|^{2(n+1)}}{[2(n+1)]!\ |x|^{2n}} = \frac{|x|^2}{(2n+2)(2n+1)} \xrightarrow[n\to\infty]{} 0$$

$\forall\, x \in \mathbb{R}$.

Logo, o raio é $R = \infty$.

(c) $f(x) = 2^x = e^{x \log 2}$

$\Rightarrow f^{(n)}(x) = (\log 2)^n e^{x\log 2} = (\log 2)^n \cdot 2^x$

$\Rightarrow f^{(n)}(0) = (\log 2)^n$

A série de Taylor é

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(\log 2)^n}{n!} \cdot x^n$$

Raio de convergência R:

- Teste da razão

$$\frac{(\log 2)^{n+1}|x|^{n+1}n!}{(n+1)!(\log 2)^n|x|^n} = \frac{(\log 2)|x|}{n+1} \xrightarrow[n\to\infty]{} 0$$

$\forall x \in \mathbb{R}$

Logo, $R = \infty$.

(d) $f(x) = \operatorname{senh} x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = 1$

Como $f''(x) = \operatorname{senh} x$ segue que

$$f^{(n)}(0) = \begin{cases} 0, & \text{se } n \text{ é par} \\ 1, & \text{se } n \text{ é ímpar} \end{cases}$$

Série de Taylor:

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

Raio de convergência R:

- Teste da razão

$$\frac{|x|^{2(n+1)+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)} \xrightarrow[n\to\infty]{} 0$$

$\forall x \in \mathbb{R}$.

Logo, $R = \infty$.

(e) $f(x) = \log(1+x)$

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{2!}{(1+x)^3}, \quad f^{(4)}(x) = \frac{-3!}{(1+x)^4}$$

Temos

$$f^{(n)}(x) = \frac{(-1)^{n+1}}{(1+x)^n} \cdot (n-1)! \qquad (n \geq 1)$$

Série de Taylor:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot x^n$$

Raio de convergência R:

- Teste da raiz

$$\frac{\sqrt[n]{|x|^n}}{\sqrt[n]{n}} = \frac{|x|}{\sqrt[n]{n}} \xrightarrow[n\to\infty]{} |x|$$

Logo, $R = 1$

Exercício 2. Encontre a série de Taylor das seguintes funções centradas no valor a dado:

(a) $f(x) = x^5 + 2x^3 + x$, $a = 2$ (b) $f(x) = \log x$, $a = 2$ (c) $f(x) = e^{2x}$, $a = 3$

(d) $f(x) = \operatorname{sen} x$, $a = \pi$ (e) $f(x) = \sqrt{x}$, $a = 16$ (f) $f(x) = 1/x^2$, $a = -3$

(a) $f(x) = x^5 + 2x^3 + x \Rightarrow f(2) = 50$

$f'(x) = 5x^4 + 6x^2 + 1 \Rightarrow f'(2) = 105$

$f''(x) = 20x^3 + 12x \Rightarrow f''(2) = 184$

$f'''(x) = 60x^2 + 12 \Rightarrow f'''(2) = 252$

$f^{(4)}(x) = 120x \Rightarrow f^{(4)}(2) = 240$

$f^{(5)}(x) = 120$

$f^{(n)}(x) = 0 \quad (n \geq 6)$

Série de Taylor em torno de $x = 2$:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= 120(x-2)^5 + 240(x-2)^4 + 252(x-2)^3$$

$$+ 184(x-2)^2 + 105(x-2) + 50$$

(b) $f(x) = \log x$

$f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2!}{x^3}$

$\Rightarrow f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \quad (n \geq 1)$

Série de Taylor em torno de $x=2$:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$= \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-2)^n$$

(c) $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$, $f''(x) = 2^2 e^{2x}$

$\Rightarrow f^{(n)}(x) = 2^n \cdot e^{2x}$

Série de Taylor em torno de $x = 3$:

$$T(x) = \sum_{n=0}^{\infty} \frac{2^n \cdot e^6}{n!}(x-3)^n$$

(d)

$$f(x) = \operatorname{sen} x \Rightarrow f(\pi) = 0$$
$$f'(x) = \cos x \Rightarrow f'(\pi) = -1$$
$$f''(x) = -\operatorname{sen} x \Rightarrow f''(\pi) = 0$$
$$f'''(x) = -\cos x \Rightarrow f'''(\pi) = 1$$

As derivadas são cíclicas: se $m \in \mathbb{N}^*$ e $k \in \{0, 1, 2, 3\}$,

$$f^{(4m+k)}(\pi) = \begin{cases} 0, & k \text{ par} \\ -1, & k = 1 \\ 1, & k = 3 \end{cases}$$

Logo:

$$f^{(2n+1)}(\pi) = (-1)^{n+1}$$

Série de Taylor em $x = \pi$:

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x - \pi)^{2n+1}$$

(e) $f(x) = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$

$\vdots$

$f^{(n)}(x) = 1/2 \cdot (1/2 - 1) \cdots [1/2 - (n-1)]\, x^{1/2 - n}$

Temos, para $16 = 2^4$,

$$f^{(n)}(2^4) = \prod_{k=0}^{n-1}\left(\frac{1}{2} - k\right)(2^4)^{1/2-n} \quad (n \geq 1)$$

$$= \prod_{k=0}^{n-1} \frac{(1-2k)}{2^n} \cdot \frac{2^2}{2^n} = \prod_{k=0}^{n-1} \frac{(1-2k)}{2^{2n-2}}$$

$$= \prod_{k=1}^{n-1} \frac{(1-2k)}{2^{2n-2}} = (-1)^{n-1} \prod_{k=1}^{n-1} \frac{(2k-1)}{2^{2n-2}} \quad (n \geq 2)$$

$$= \frac{(-1)^{n-1}}{2^{2(n-1)}} \cdot \frac{(2n-3)!}{2^{n-1} \cdot (n-1)!} \quad (n \geq 2)$$

Série de Taylor em $x = 16$:

$$T(x) = 4 + \frac{1}{8} + \sum_{n=2}^{\infty} \left(\frac{-1}{8}\right)^{n-1} \cdot \frac{(2n-3)!}{n!(n-1)!} (x-16)^n$$

OBS: $\prod_{k=0}^{n} a_k = a_0 \cdot a_1 \cdot a_2 \cdots a_n$

(d) $f(x) = x^{-2} \Rightarrow f^{(n)}(x) = \prod_{k=0}^{n-1} (-2-k)\, x^{-2-n}$ $(n \geq 1)$

Série de Taylor em $x = -3$:

$$T(x) = \frac{1}{9} + \sum_{n=1}^{\infty} (-1)^n \prod_{k=0}^{n-1} (2+k) (-1)^{n+2} \cdot 3^{-(n+2)} \cdot \frac{(x+3)^n}{n!}$$

$$= \frac{1}{9} + \sum_{n=1}^{\infty} \frac{(n+1)!}{n!} \cdot \frac{1}{3^{n+2}} \cdot (x+3)^n$$

$$= \frac{1}{9} + \sum_{n=1}^{\infty} \frac{(n+1)}{3^{n+2}} (x+3)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)}{3^{n+2}} (x+3)^n$$

Exercício 3. Use a série binomial para expandir a função como série de potências. Diga qual é o raio de convergência.

(a) $\sqrt[4]{1-x}$ (b) $\dfrac{1}{(2+x)^3}$ (c) $\sqrt[3]{8+x}$ (d) $(1-x)^{3/4}$

(a) $f(x) = (1-x)^{1/4} = [1 + (-x)]^{1/4}$

A série binomial diz que, se $|x|<1$,

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$$

Logo:

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\cdots\left[\frac{1}{4}-(n-1)\right]}{n!}(-x)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n} \cdot \frac{[4(n-1)-1][4(n-2)-1]\cdots[4-1]}{n!}(-1)^n x^n$$

$$= 1 - \frac{x}{4} - \sum_{n=2}^{\infty} \frac{[4(n-1)-1][4(n-2)-1]\cdots[4-1]}{4^n \cdot n!} x^n$$

O raio de convergência é $|-x| < 1$

$\Rightarrow R = 1.$

(b) $f(x) = (2+x)^{-3} = 2^{-3}\cdot(1 + x/2)^{-3}$

Usando a série binomial, vem que

$$f(x) = \frac{1}{8}\left[1 + \sum_{n=1}^{\infty} \frac{(-3)(-3-1)\cdots[-3-(n-1)]}{n!}\left(\frac{x}{2}\right)^n\right]$$

$$= \frac{1}{8} + \sum_{n=1}^{\infty}(-1)^n \cdot \frac{3\cdot(3+1)\cdots\overbrace{[3+(n-1)]}^{n+2}}{n!\,2^{n+3}}\, x^n$$

$$= \frac{1}{8} + \sum_{n=1}^{\infty}(-1)^n \cdot \frac{(n+2)!}{2^{n+4}\, n!}\, x^n$$

$$= \sum_{n=0}^{\infty}(-1)^n \cdot \frac{(n+2)!}{2^{n+4}\cdot n!}\, x^n$$

Raio de convergência R: $|x/2| < 1$

$\Rightarrow |x| < 2$

$\Rightarrow R = 2$

(c) $f(x) = (8+x)^{1/3} = 2(1 + x/8)^{1/3}$

Usando a série binomial,

$$f(x) = 2\left\{1 + \sum_{n=1}^{\infty} \frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdots\left[\frac{1}{3}-(n-1)\right]}{n!}\left(\frac{x}{8}\right)^n\right\}$$

$$= 2\left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \frac{(1-3)(1-2\cdot 3)\cdots[1-(n-1)\cdot 3]}{n!}\left(\frac{x}{8}\right)^n \right\}$$

$$= 2\left\{ 1 + \frac{1}{3}\left(\frac{x}{8}\right) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{3^n \cdot n!}\left[\prod_{k=1}^{n-1}(3k-1)\right]\left(\frac{x}{8}\right)^n \right\}$$

Raio de convergência R: $|x/8| < 1$

$\Rightarrow |x| < 8$

$\Rightarrow R = 8$

(d) $f(x) = (1-x)^{3/4} = [1 + (-x)]^{3/4}$

Usando a série binomial,

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdots\left[\frac{3}{4}-(n-1)\right]}{n!}(-x)^n$$

$$= 1 - \frac{3}{4}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n!}\cdot\frac{3}{4^n}\left[\prod_{k=1}^{n-1}(4k-3)\right](-1)^n \cdot x^n$$

$$= 1 - \frac{3}{4}x - \sum_{n=2}^{\infty} 3\cdot\left[\prod_{k=1}^{n-1}(4k-3)\right]\frac{x^n}{n!\,4^n}$$

O raio de convergência é $|-x| < 1$

$\Rightarrow R = 1.$

Exercício 4. Obtenha a série de Maclaurin das seguintes funções:

(a) $f(x) = \arctan x^2$ (b) $f(x) = x\cos 2x$ (c) $f(x) = \dfrac{x}{\sqrt{4+x^2}}$ (d) $f(x) = \operatorname{sen}^2 x$

(a) Sabemos que

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad (|x|<1)$$

$$\Rightarrow \arctan x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2} \quad (|x|<1)$$

(b) Sabemos que

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (\forall x \in \mathbb{R})$$

$$\Rightarrow x\cos 2x = x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 4^n \cdot x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n+1} \quad (\forall x \in \mathbb{R})$$

(c) $f(x) = \dfrac{x}{\sqrt{4+x^2}}$

Repare que $\dfrac{d}{dx}(4+x^2)^{1/2} = \dfrac{1}{2} \cdot 2x \cdot (4+x^2)^{-1/2} = f(x)$

Usando a série binomial,

$$(4+x^2)^{1/2} = 2\left[1+(x/2)^2\right]^{1/2}$$

$$= 2\cdot\left\{1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdots\left[\frac{1}{2}-(n-1)\right]}{n!}\left(\frac{x}{2}\right)^{2n}\right\}$$

$$= 2\cdot\left\{1+\frac{1}{2}\left(\frac{x}{2}\right)^2+\right.$$

$$\left.+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{2^n\cdot n!}\cdot[2(n-1)-1][2(n-2)-1]\cdots[2-1]\frac{x^{2n}}{2^{2n}}\right\}$$

$$= 2\cdot\left\{1+\frac{1}{2}\left(\frac{x}{2}\right)^2+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{2^n\cdot n!}\cdot\frac{[2(n-1)-1]!}{2^{n-2}(n-2)!}\cdot\frac{x^{2n}}{2^{2n}}\right\}$$

$$= 2+\frac{x^2}{4}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{2^{4n-3}}\cdot\frac{[2(n-1)-1]!}{n!\,(n-2)!}\,x^{2n}$$

(d) $f(x) = \operatorname{sen}^2 x = \dfrac{1-\cos 2x}{2}$

Temos, pela série de Taylor do cosseno,

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \qquad (\forall x \in \mathbb{R})$$

Logo

$$f(x) = \frac{1}{2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}4^n\cdot x^n = \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 2^{2n-1}}{(2n)!}x^n$$

Exercício 5. Use a fórmula

$$\operatorname{ar}\tanh x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

e a série de Taylor de $\log(1+x)$ para mostrar que

$$\operatorname{ar}\tanh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

Lembre que

$$\log(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n \cdot x^{n+1}}{n+1} \qquad (|x|<1)$$

$$\Rightarrow \log(1-x) = \sum_{n=0}^{\infty}\frac{(-1)^n \cdot (-x)^{n+1}}{n+1} = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

Daí,

$$\operatorname{artanh} x = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n+1}{2(n+1)}x^{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} \qquad (|x|<1)$$

Exercício 6. Calcule a integral como uma série infinita:

(a) $\int \sqrt{1+x^3}\,dx$ (b) $\int \frac{\cos x - 1}{x}\,dx$ (c) $\int x^2 \operatorname{sen} x^2\,dx$

(d) $\int \arctan x^2\,dx$ (e) $\int x^2 e^{-x^2}\,dx$

(a) A série binomial diz que

$$(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdots\left[\frac{1}{2}-(n-1)\right]}{n!} x^n$$

$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{[2(n-1)-1][2(n-2)-1]\cdots[2-1]}{2^n\, n!} x^n$$

$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{2^n} \cdot \frac{[2(n-1)-1]!}{n!\,(n-1)!\,2^{n-1}} \cdot x^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,[2(n-1)-1]!}{2^{2n-1}\cdot n!\,(n-1)!} \cdot x^n$$

Logo,

$$\int (1+x^3)^{1/2}\,dx = \int 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,[2(n-1)-1]!}{2^{2n-1}\cdot n!\,(n-1)!} \cdot x^n\,dx$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,[2(n-1)-1]!}{2^{2n-1}\cdot (n+1)!\,(n-1)!} \cdot x^{n+1}$$

(b) $$\int \frac{\cos x - 1}{x}\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n \cdot (2n)!}$$

(c) $$\int x^2 \operatorname{sen} x^2\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n [x^2]^{2n+1} \cdot x^2}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{4(n+1)}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(4n+5)\cdot(2n+1)!}$$

(d) $$\int \arctan x^2\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+2)}$$

(e) $$\int x^2 e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+3}}{n!\,(2n+3)}$$

Exercício 7. Encontre a soma da série:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n5^n}$ (c) $1 - \log 2 + \dfrac{(\log 2)^2}{2!} - \dfrac{(\log 2)^3}{3!} + \ldots$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$ (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$ (f) $\dfrac{1}{1 \cdot 2} - \dfrac{1}{3 \cdot 2^3} + \dfrac{1}{5 \cdot 2^5} - \dfrac{1}{7 \cdot 2^7} + \ldots$

(g) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$

(a) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1}$$

(b) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{3}{5}\right)^n = \log \frac{3}{5}$$

(c) $$\sum_{n=0}^{\infty} \frac{(-\log 2)^n}{n!} = e^{-\log 2} = \frac{1}{2}$$

(d) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} = \operatorname{sen} \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

(e) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{6}\right)^{2n} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

(f) Note que

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \int \frac{1}{1-x^2} dx$$

$$= \int \frac{1}{1-x} \cdot \frac{1}{1+x}\, dx = \frac{1}{2} \int \frac{1}{1-x} + \frac{1}{1+x}\, dx$$

$$= \frac{1}{2}\left[-\log(1-x) + \log(1+x)\right]$$

$$= \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{1}{2^{2n+1}} = \frac{1}{2} \log\left(\frac{3/2}{1/2}\right) = \frac{1}{2} \log 3$$

(g) $$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \frac{1}{(x+2)^3} \cdot \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}$$

$$= \frac{1}{(x+2)^3} \cdot \sum_{n=3}^{\infty} \frac{(x+2)^n}{n!}$$

$$= \frac{1}{(x+2)^3} \cdot \left[\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} - \sum_{n=0}^{2} \frac{(x+2)^n}{n!}\right]$$

$$= \frac{e^{x+2} - 1 - (x+2) - (x+2)^2/2}{(x+2)^3}$$

Exercício 8. Encontre a série de Taylor de $\operatorname{arcsen} x$ em torno de $x = 0$ usando

$$\operatorname{arcsen} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

Seja $f(x) = (1-x^2)^{-1/2}$. Usando a expansão binomial, obtemos

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(n-1)\right]}{n!}(-x^2)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \cdot \frac{[2(n-1)-1][2(n-2)-1]\cdots[2-1]}{n!}(-1)^n \cdot x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{[2(n-1)-1]!}{n!\,(n-2)!\,2^n} \cdot x^{2n}$$

Daí,

$$\operatorname{arcsen} x = \int f(x)\,dx$$

$$= x + \sum_{n=1}^{\infty} \frac{[2(n-1)-1]!\; x^{2n+1}}{n!\,(n-2)!\,2^n\,(2n+1)}$$

Exercício 9. Encontre as séries de Taylor das seguintes funções em torno de $x = 0$:

(a) $\operatorname{ar senh} x$ (b) $\int_0^x e^{-t^2}\, dt$ (c) $\int_0^x \frac{\operatorname{sen} t}{t}\, dt$

(a) $\operatorname{arsenh} x = \int \frac{dx}{\sqrt{x^2+1}}$

Pela série binomial,

$$(1+x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(n-1)\right]}{n!}(x^2)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \cdot \frac{[2(n-1)-1][2(n-2)-1]\cdots[2-1]}{n!} \cdot x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{[2(n-1)-1]!}{n!\,(n-2)!\,2^n} \cdot x^{2n}$$

Logo, integrando termo a termo,

$$\operatorname{arsenh} x = x + \sum_{n=1}^{\infty} \frac{(-1)^n [2(n-1)-1]!}{n!\,(n-2)!\,2^n \cdot (2n+1)} \cdot x^{2n+1}$$

(b) Pela série exponencial,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Integrando termo a termo, vem

$$\int_0^x e^{-t^2}dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}$$

(c) Da série para o seno, temos

$$\operatorname{sen} t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \frac{\operatorname{sen} t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

Integrando termo a termo, vem

$$\int_0^x \frac{\operatorname{sen} t}{t} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)^2 \cdot (2n)!}$$

Exercício 10.

(a) Mostre que a série de Taylor da função

$$f(x) = \frac{x}{1 - x - x^2} \quad \text{é} \quad \sum_{n=0}^{\infty} f_n x^n,$$

onde (f_n) é a sequência de Fibonacci, dada por $f_0 = 0$, $f_1 = 1$ e $f_n = f_{n-1} + f_{n-2}$ $(n \geqslant 2)$. (*Sugestão: escreva* $x/(1 - x - x^2) = c_0 + c_1 x + c_2 x^2 + \ldots$ *e multiplique os dois lados por* $(1 - x - x^2)$)

(b) Escreva $f(x)$ na forma de soma de frações parciais para encontrar a série de Taylor de uma forma diferente, obtendo uma fórmula explícita para f_n em função de n.

(a) $$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} c_n x^n$$

$$\Rightarrow \quad x = \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} c_{n-1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n$$

$$= c_0 + c_1 x - c_0 x + \sum_{n=2}^{\infty} (c_n - c_{n-1} - c_{n-2}) x^n$$

Logo,

$c_0 = 0$, $c_1 = 1$ e $c_n - c_{n-1} - c_{n-2} = 0$ $(n \geqslant 2)$

Essa é a sequência de Fibonacci:
$(c_n = f_n)$

$f_0 = f_1 = 1$, $\quad f_n = f_{n-1} + f_{n-2}$ $(n \geqslant 2)$

Portanto,

$$f(x) = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} f_n x^n$$

(b) As raízes de x^2+x-1 são

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

Logo,

$$\frac{x}{1-x-x^2} = \frac{-x}{\left(x+\frac{1-\sqrt{5}}{2}\right)\left(x+\frac{1+\sqrt{5}}{2}\right)}$$

$$= \frac{a}{x+\frac{1-\sqrt{5}}{2}} + \frac{b}{x+\frac{1+\sqrt{5}}{2}}$$

$$= \frac{(a+b)x + a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right)}{\left(x+\frac{1-\sqrt{5}}{2}\right)\left(x+\frac{1+\sqrt{5}}{2}\right)}$$

Daí,

$$\begin{cases} a+b=-1 \Rightarrow b=-(a+1) \\ a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 0 \end{cases}$$

$$\Rightarrow a + a\sqrt{5} - a + a\sqrt{5} = 1-\sqrt{5}$$

$$\Rightarrow a = \frac{1-\sqrt{5}}{2\sqrt{5}}, \quad b = \frac{\sqrt{5}-1}{2\sqrt{5}} - \frac{2\sqrt{5}}{2\sqrt{5}}$$

$$\Rightarrow b = -\frac{(1+\sqrt{5})}{2\sqrt{5}}$$

Assim,

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \, \frac{\frac{1-\sqrt{5}}{2}}{\frac{2x}{2} + \frac{1-\sqrt{5}}{2}} - \frac{1}{\sqrt{5}} \, \frac{\frac{1+\sqrt{5}}{2}}{\frac{2x}{2} + \frac{1+\sqrt{5}}{2}}$$

$$= \frac{1}{\sqrt{5}} \, \frac{1}{\frac{2x}{1-\sqrt{5}} + 1} - \frac{1}{\sqrt{5}} \cdot \frac{1}{\frac{2x}{1+\sqrt{5}} + 1}$$

$$\frac{2}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = -\frac{(1+\sqrt{5})}{2} \qquad \frac{2}{1+\sqrt{5}} \cdot \frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{\sqrt{5}-1}{2}$$

$$= \frac{1}{\sqrt{5}} \, \frac{1}{1 - \left(\frac{1+\sqrt{5}}{2}\right)x} - \frac{1}{\sqrt{5}} \, \frac{1}{1 - \left(\frac{1-\sqrt{5}}{2}\right)x}$$

$$= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] x^n$$

Assim,

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$