LOGARITMOS

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DEFINIÇÃO

Se $a, b \in \mathbb{R}_+$ com $a \neq 1$, o logaritmo de b na base a é o expoente que se deve colocar na base a para que a potência obtida seja b.

$$\log_a b = x \longleftrightarrow a^x = b$$

a : base

x : logaritmo

b : logaritmando ou antilogaritmo

EXEMPLOS

a) $\log_2 8 = x \quad \therefore \quad 2^x = 8$

$2^x = 2^3$

$\log_2 8 = 3$

$\boxed{x = 3}$

b) $\log_{10} 100 = x \quad \therefore \quad 10^x = 100$

$10^x = 10^2$

$\boxed{x = 2}$

$\log_{10} 100 = 2$

c) $\log_9 1 = x \quad \therefore \quad 9^x = 1$

$\boxed{x = 0}$

$\log_9 1 = 0$

d) $\log_{12} 12 = 1 \quad \therefore \quad 12^1 = 12$

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e) $\log_{0,2} 25 = x$ $\therefore$ $0{,}2^{x} = 25$

$\left(\frac{1}{5}\right)^{x} = 5^{2}$

$\log_{0,2} 25 = -2$

$(5^{-1})^{x} = 5^{2}$

$5^{-x} = 5^{2}$

$\boxed{x = -2}$

f) $\log_{0,5} 16 = x$ $\therefore$ $\left(\frac{1}{2}\right)^{x} = 16$

$(2^{-1})^{x} = 2^{4}$

$\log_{0,5} 16 = -4$

$\boxed{x = -4}$

g) $\log_{16} 0{,}5 = x$ $\therefore$ $16^{x} = \frac{1}{2}$

$2^{4x} = 2^{-1}$

$\boxed{x = -\frac{1}{4}}$

OBSERVAÇÕES

(i) $\log a = \log_{10} a$

(ii) $\log_a b$

$b \in \mathbb{R}_+^*$ $(b > 0)$

$a \in \mathbb{R}_+^* - \{1\}$ $(a > 0 \text{ e } a \neq 1)$

CONSEQUÊNCIAS

(i) $\log_a 1 = 0 \quad \therefore \quad a^0 = 1$

(ii) $\log_a a = 1 \quad \therefore \quad a^1 = a$

(iii) $\log_a a^x = x \quad \therefore \quad a^x = a^x$

(iv) $a^{\log_a b} = b$

$\log_a b = z \quad \therefore \quad a^z = b$

$a^{\log_a b} = a^z = b$

(v) $\log_a b = \log_a c \longleftrightarrow b = c$

$\log_a b = x \quad \therefore \quad a^x = b$

$\log_a c = y \quad \therefore \quad a^y = c$

$\log_a b = \log_a c$

$x = y \longrightarrow a^x = a^y$

$$\downarrow$$

$$\boxed{b = c}$$

EXEMPLOS

(i) $\log_{3^{-1}} \sqrt[5]{81} = x \quad \therefore \quad (3^{-1})^{x} = \sqrt[5]{3^4}$

$3^{-x} = (3^4)^{1/5} \quad \therefore \quad 3^{-x} = 3^{4/5} \quad \therefore \quad \boxed{x = -4/5}$

$$\boxed{\log_{1/3} \sqrt[5]{81} = -4/5}$$

(ii) $5^{-1+\log_5 2} = \ ?$

$5^{-1+\log_5 2} = 5^{-1} \cdot 5^{\log_5 2} = \frac{1}{5} \cdot 2 = 2/5$

(iii) Determine x tal que $\log_x 10x = 2$

$\log_x 10x = 2 \quad \therefore \quad x^2 = 10x \quad \therefore \quad x^2 - 10x = 0$

$x(x-10) = 0$

$x = 10$ ✓

$x = 0$ ✗

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PROPRIEDADES DO LOGARITMO

#01 . PRODUTO

$$\boxed{\log_a (b \cdot c) = \log_a b + \log_a c}$$

Prova :

$$\left.\begin{array}{l} \log_a b = x \therefore b = a^x \\ \log_a c = y \therefore c = a^y \end{array}\right\} (*)$$

$$(*)\ \log_a (b \cdot c) = \log_a (a^x \cdot a^y) = \log_a a^{x+y} = x + y =$$

$$= \log_a b + \log_a c$$

Exemplo :

(i) $\log_6 12 = \log_6 (6 \cdot 2) = \log_6 6 + \log_6 2 = 1 + \log_6 2$

(ii) $\log 7 + \log 5 = \log 35$

#02 . QUOCIENTE

$$\boxed{\log_a(b/c) = \log_a b - \log_a c}$$

Prova:

$$\left.\begin{array}{l} \log_a b = x \therefore b = a^x \\ \log_a c = y \therefore c = a^y \end{array}\right\} (*)$$

$$(*) \log_a(b/c) = \log_a(a^x/a^y) = \log_a a^{x-y} = x - y$$

$$= \log_a b - \log_a c$$

Exemplo:

(i) $\log(10/13) = \log 10 - \log 13$

(ii) $\log_2 10 - \log_2 5 = \log_2(10/5) = \log_2 2 = 1$

#03 . EXPOENTE

$$\boxed{\log_a b^x = x \cdot \log_a b}$$

Prova :

$$\log_a b = \alpha \quad \therefore \quad b = a^{\alpha} \dashrightarrow b^x = a^{\alpha \cdot x}$$

$$\log_a b^x = \beta \quad \therefore \quad b^x = a^{\beta} \dashrightarrow b^x = a^{\beta}$$

$$a^{\alpha \cdot x} = a^{\beta} \quad \therefore \quad \alpha \cdot x = \beta$$

$$x \cdot \log_a b = \log_a b^x$$

Exemplo :

(i) $\log_2 5^3 = 3 \cdot \log_2 5$

(ii) $2 \cdot \log_3 5 = \log_3 5^2$

$$\boxed{\log_{a^x} b = \frac{1}{x} \cdot \log_a b}$$

Prova:

$$\left.\begin{aligned} \log_a b = \alpha \quad \therefore \quad b = a^{\alpha} \\ \log_{a^x} b = \beta \quad \therefore \quad b = a^{x\beta} \end{aligned}\right\} \quad a^{\alpha} = a^{x\beta}$$

$$\alpha = x \cdot \beta \quad \therefore \quad \beta = \frac{1}{x} \cdot \alpha \quad \therefore \quad \log_{a^x} b = \frac{1}{x} \log_a b$$

Exemplo:

(i) $\log_{27} 5 = \log_{3^3} 5 = \frac{1}{3} \cdot \log_3 5$

(ii) $\log_{32} 10 = \log_{2^5} 10 = \frac{1}{5} \cdot \log_2 10$

(iii) $\frac{1}{7} \cdot \log_{16} 5 = \log_{16^7} 5 = \log_{16} 5^{1/7}$

EXEMPLO 01

Resolva a equação

$$\log_3(x-1) + \log_{3^2}(x-1)^2 - \log_3 x = -1$$

R:

$$\log_3(x-1) + \frac{2}{2}\cdot\log_{3^2}(x-1)^2 - \log_3 x = -1$$

$$\log_3(x-1) + \log_3(x-1) - \log_3 x = -1$$

$$2\cdot\log_3(x-1) - \log_3 x = -1 \therefore \log_3(x-1)^2 - \log_3 x = -1$$

$$\log_3\left[\frac{(x-1)^2}{x}\right] = -1 \therefore 3^{-1} = \frac{(x-1)^2}{x} = \frac{1}{3}$$

$$3(x-1)^2 = x \therefore 3(x^2-2x+1) = x$$

$$3x^2 - 6x + 3 = x$$

$$3x^2 - 7x + 3 = 0$$

$$\Delta = 49 - 4 \cdot 3 \cdot 3 = 49 - 36 = 13$$

$$x = \frac{7 \pm \sqrt{13}}{6}$$

$$\sqrt{9} = 3$$
$$\sqrt{13} = 3{,}2$$
$$\sqrt{16} = 4$$

CUIDADO :

i) $x > 0$ — x_1 ✓ , x_2 ✓

ii) $x - 1 > 0$ — x_1 ✓ , x_2 ✗

Solução :

$$\boxed{x = \frac{7 + \sqrt{13}}{6}}$$

EXEMPLO 02

Resolva a equação

$$2\cdot\log(1/2) + \log 4 - \log\left(\frac{1}{10x}\right) = 3\log x^2$$

R:

$$2\cdot\log(2^{-1}) + \log 2^2 - \log(10x)^{-1} = 3\log x^2$$

$$-2\cancel{\log 2} + 2\cdot\cancel{\log 2} + \log(10x) = \log(x^2)^3$$

$$\log(10x) = \log x^6 \quad \therefore \quad x^6 = 10x$$

i) $\cancel{x = 0}$

ii) $x = \sqrt[5]{10} = 10^{1/5}$

$$R: \; x = \sqrt[5]{10}$$

EXEMPLO 03

Dado $\log_3 2 = 0,63$ calcule $\log_3 6$

R:

$$\log_3 6 = \log_3 (2 \cdot 3) = \log_3 2 + \log_3 3$$

$$\log_3 6 = \log_3 2 + 1$$

$$= 0,63 + 1$$

$$= \underline{1,63}$$

COLOGARITMO

Seja $\log_3 2 = a$

$\hookrightarrow \log_{1/3} 2 = \log_{3^{-1}} 2 = \frac{1}{-1} \cdot \log_3 2 = -\log_3 2 = -a$

$\hookrightarrow \log_3 (1/2) = \log_3 (2^{-1}) = -\log_3 2 = -a$

→ Ao inverter a base ou o logaritmando o logaritmo troca de sinal:

$$\boxed{\log_b a^{-1} = \log_{b^{-1}} a = -\log_b a = \text{colog}_b\, a}$$

MUDANÇA DE BASE

$$\boxed{\log_a b = \frac{\log_c b}{\log_c a}}$$

EXEMPLO:

(i) Dado $\log 2 = 0{,}3$ e $\log 7 = 0{,}84$ calcule $\log_2 7$:

$$\log_2 7 = \frac{\log_c 7}{\log_c 2} = \frac{\log_{10} 7}{\log_{10} 2} = \frac{0{,}84}{0{,}3} = 2{,}8 \;//$$

(ii) Dado $\log_2 7 = a$, calcule $\log_{14} 8$.

$$\log_{14} 8 = \frac{\log_c 8}{\log_c 14} = \frac{\log_2 8}{\log_2 14} = \frac{3}{\log_2 7 + \log_2 2}$$

$$= \frac{3}{(a+1)} \;//$$

Demonstração:

$$\left.\begin{array}{l} \log_a b = x \quad \therefore \quad b = a^x \\[2em] \log_c b = y \quad \therefore \quad b = c^y \end{array}\right\} (*)$$

$$(*) \quad a^x = c^y \quad \therefore \quad \log_c a^x = \log_c c^y$$

$$\log_c a^x = \log_c c^y \quad \therefore \quad x \cdot \log_c a = y \cdot \cancelto{1}{\log_c c}$$

$$x \cdot \log_c a = y \quad \therefore \quad \log_a b \cdot \log_c a = \log_c b$$

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad \square$$

PROPRIEDADES

(i) Inversão

$$\log_a b = \frac{\log_c b}{\log_c a} \quad \therefore \quad \log_a b = \frac{\log_b b}{\log_b a}$$

$$\boxed{\log_a b = \frac{1}{\log_b a}}$$

(ii) Produto de logaritmos

$$\log_a b = \frac{\log_c b}{\log_c a} \quad \therefore \quad \log_c b = \log_{\cancel{a}} b \cdot \log_c \cancel{a}$$

EXEMPLO: $\log_{\cancel{3}} 7 \cdot \log_{\cancel{5}} \cancel{3} \cdot \log_8 \cancel{5} = \log_8 7$

LOGARITMO NATURAL

$e \cong 2{,}71828\ldots$

NÚMERO DE EULER

$$\boxed{\ln x = \log_e x}$$

- $\ln x$ é o logaritmo natural de x ou logaritmo neperiano de x.

John Napier (1550 - 1617)

"Uma descrição da maravilhosa regra dos logaritmos."

EXEMPLO

- decaimento radioativo: $M(t) = M_0 \cdot e^{-\alpha t}$