

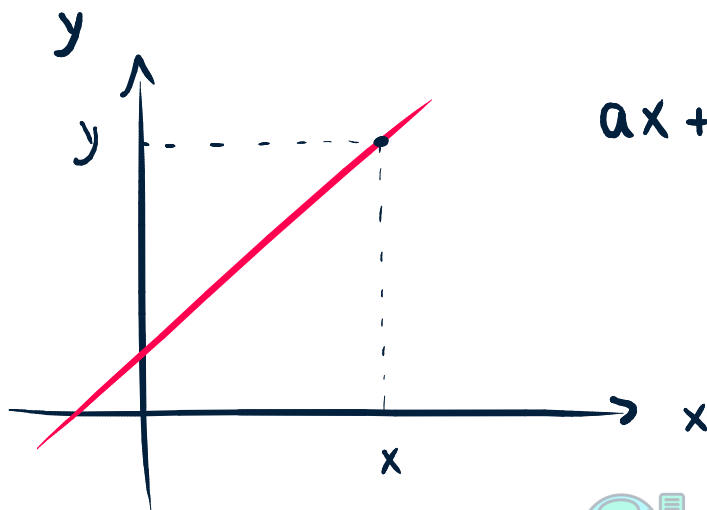
• RETAS •

• EQUAÇÃO GERAL DA RETA •

As coordenadas (x, y) de todos os pontos de uma reta no plano estão relacionadas por meio da equação :

$$a \cdot x + b \cdot y + c = 0$$

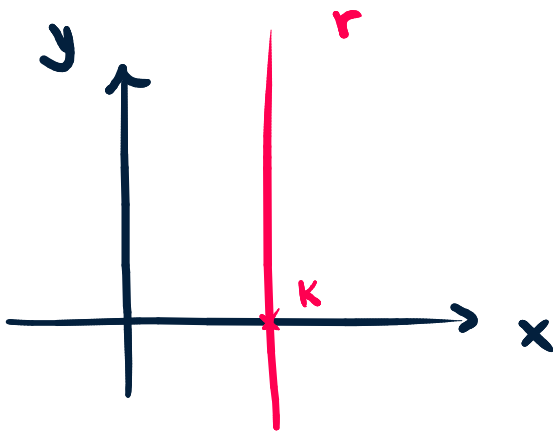
com $a, b, c \in \mathbb{R}$.



$$ax + by + c = 0$$



CASO 01:

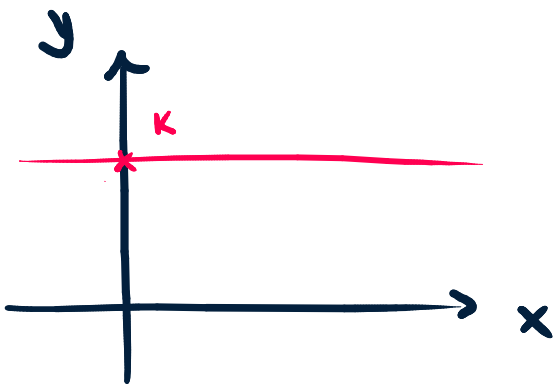


$$ax + by + c = 0$$

$$\hookrightarrow \boxed{b=0} \therefore ax + c = 0$$

$$x = -\frac{c}{a} = K$$

CASO 02:

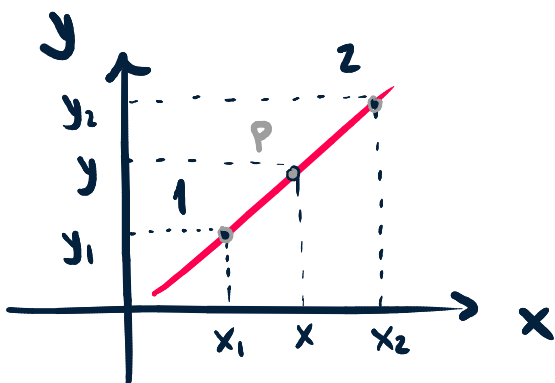


$$ax + by + c = 0$$

$$\hookrightarrow \boxed{a=0} \therefore by + c = 0$$

$$y = -\frac{c}{b} = K$$

CASO 03:



$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$P = (x, y)$$



$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$x_1 \cdot y_2 + y_1 \cdot x + x_2 \cdot y - y_2 \cdot x - x_1 \cdot y - y_1 x_2 = 0$$

$$x(\overbrace{y_1 - y_2}^a) + y(\overbrace{x_2 - x_1}^b) + (\overbrace{x_1 \cdot y_2 - y_1 x_2}^c) = 0$$

$$a \cdot x + b \cdot y + c = 0 \quad \square$$



• EQUAÇÃO REDUZIDA DA RETA •

$$ax + by + c = 0$$

→ Se a reta não é paralela ao eixo y ($b \neq 0$):

$$by = -ax - c$$

$$y = -\frac{a}{b}x - \frac{c}{b}$$

($\div b$)

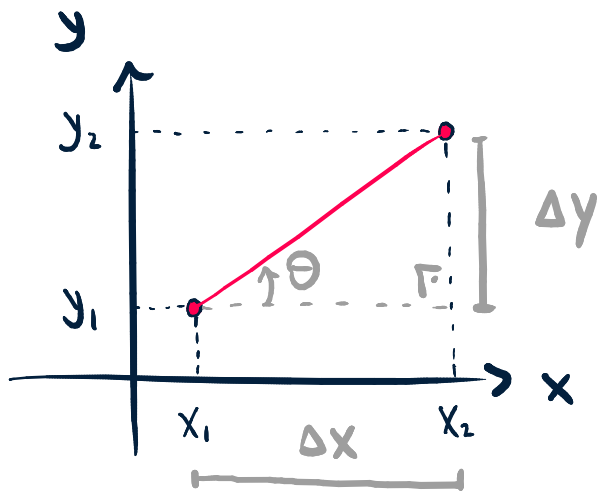
$$y = mx + n$$

• m : coeficiente angular

• n : coeficiente linear

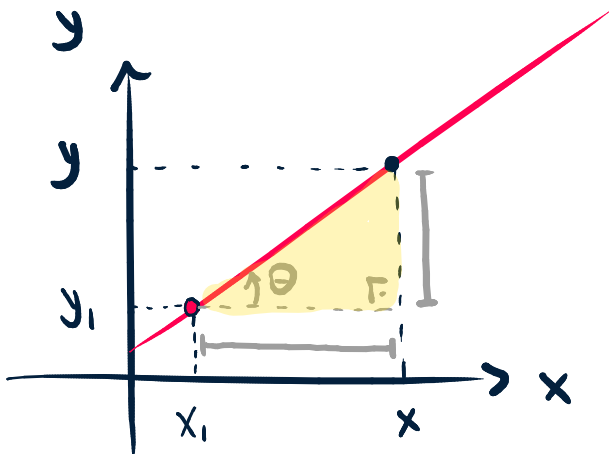


COEFICIENTE ANGULAR (m)



$$y = m \cdot x + n$$

$$m = \tan \theta = \frac{\Delta y}{\Delta x}$$



$$\tan \theta = \frac{y - y_1}{x - x_1}$$

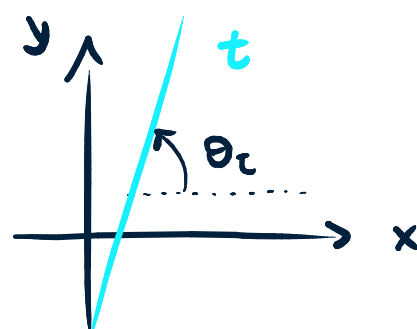
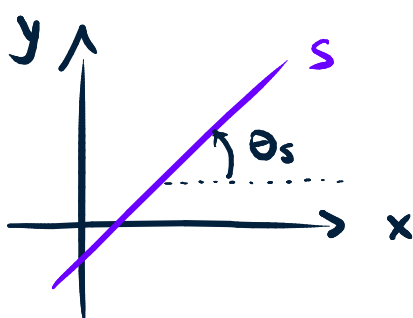
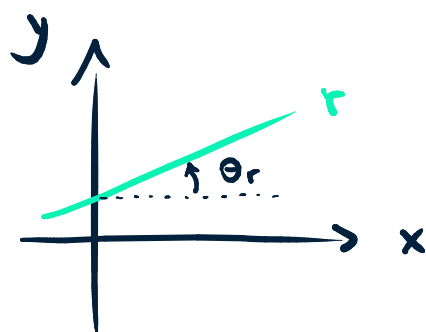
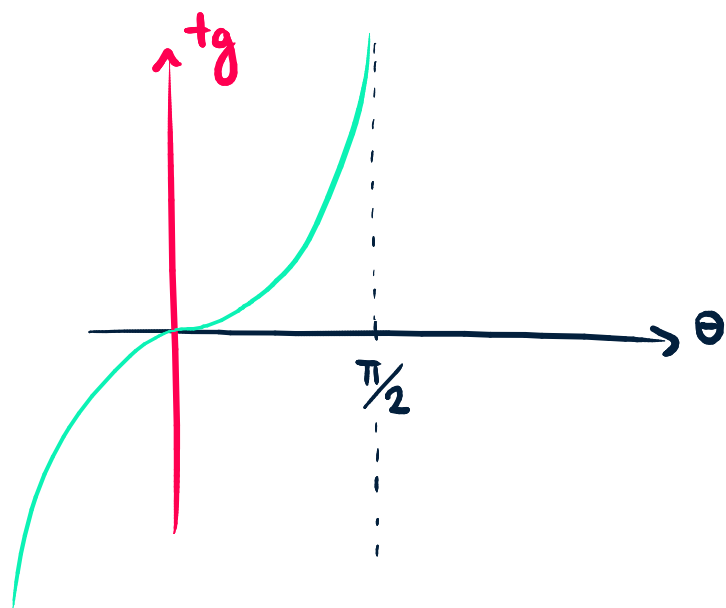
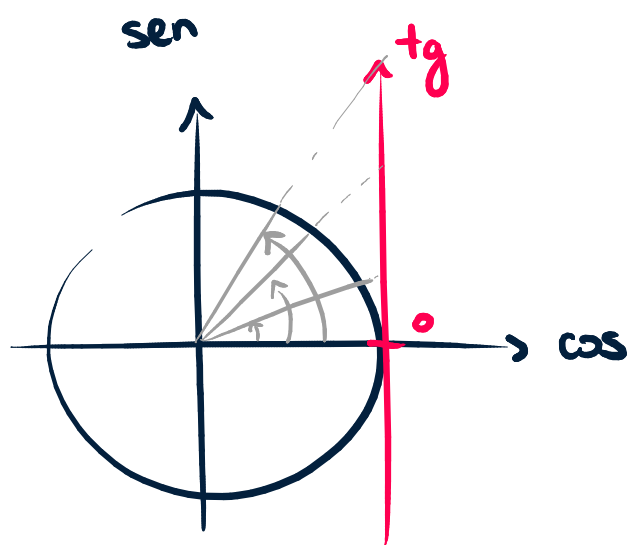
$$y - y_1 = \tan \theta \cdot x - \tan \theta x_1$$

$$y = \tan \theta \cdot x + (y_1 - \tan \theta x_1)$$

$$y = m \cdot x + n$$

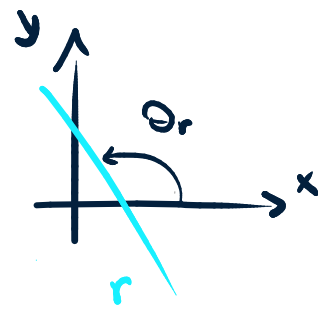
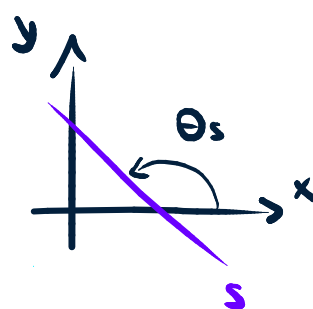
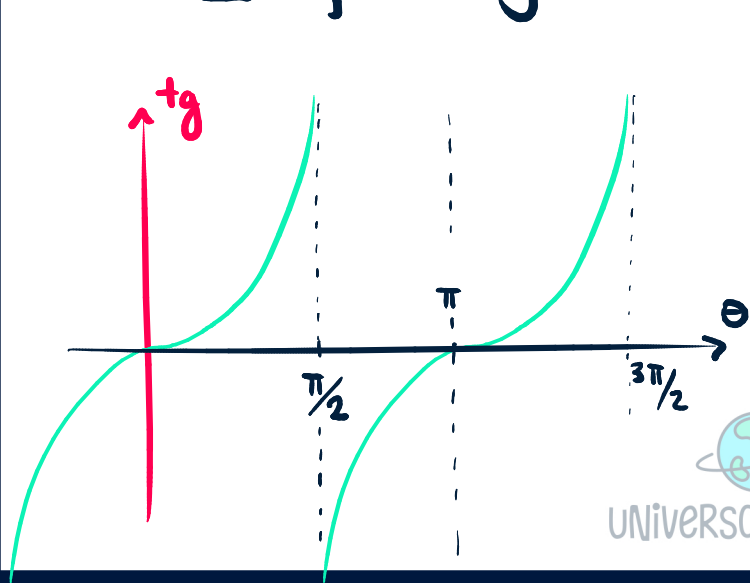


Quanto maior o valor de m , maior será a inclinação da reta:



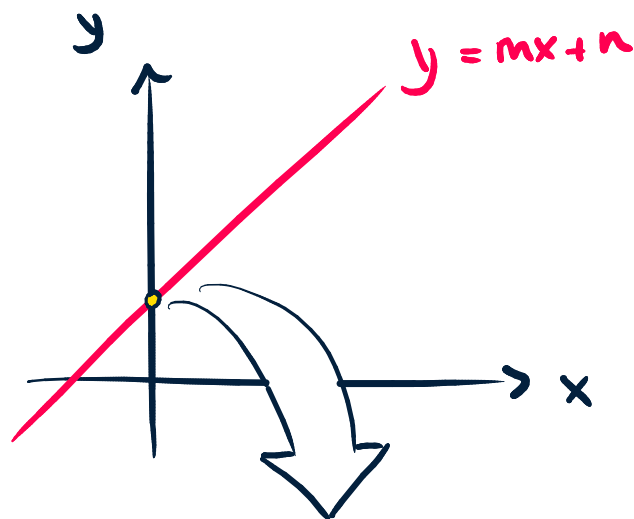
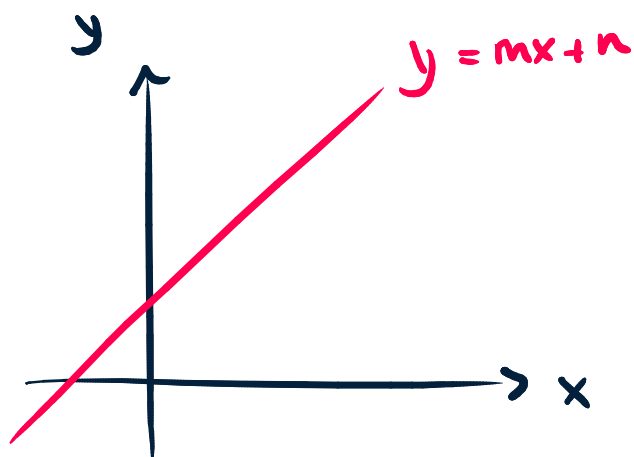
$$m_r < m_s < m_t$$

Se m for negativo a reta será decrescente:

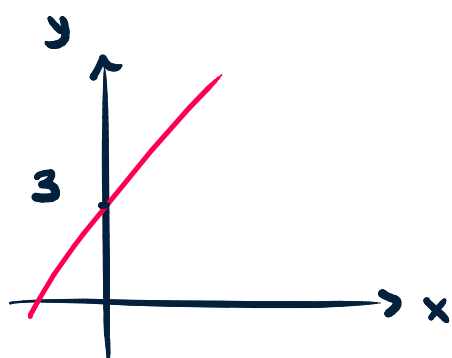


COEFICIENTE LINEAR (n)

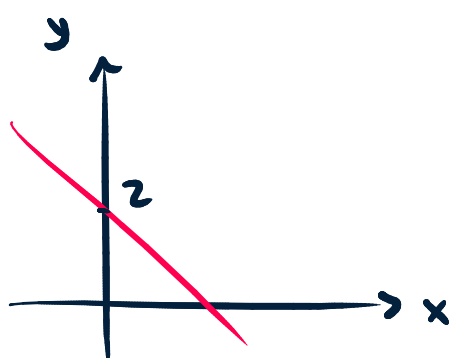
mede o local em que a reta intercepta o eixo vertical :



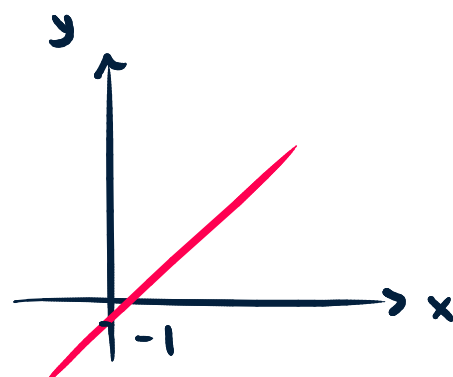
$$x = 0$$
$$y = m \cdot x + n$$
$$y = n$$



$$y = mx + 3$$



$$y = mx + 2$$



$$y = mx - 1$$

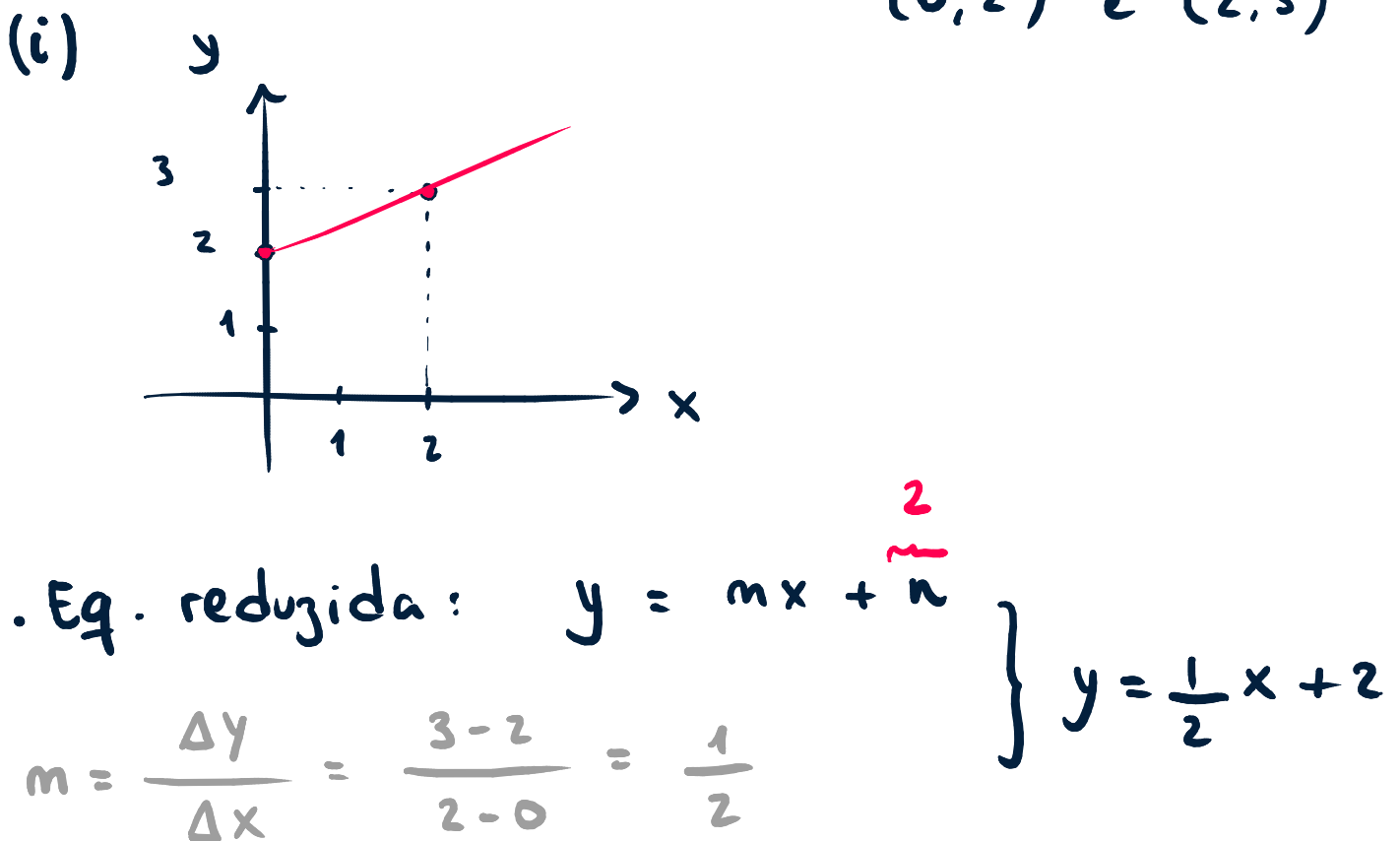


EXEMPLO 01

Obtenha a equação geral da reta que passa pelos pontos $(0, 2)$ e $(2, 3)$.

R :

SOLUÇÃO 01

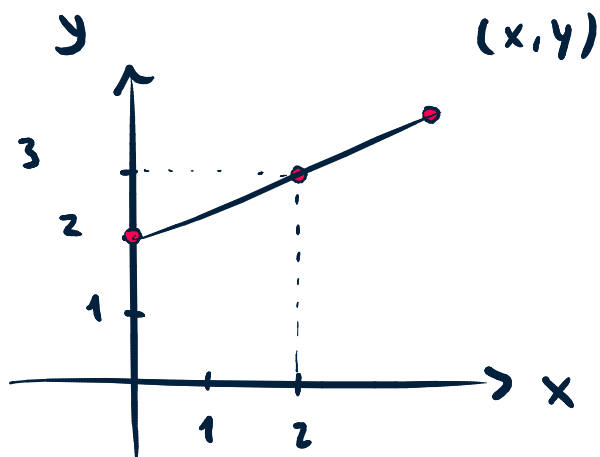


$$2y = x + 4 \quad \therefore$$

$$\boxed{x - 2y + 4 = 0}$$



SOLUÇÃO 02



$$\begin{vmatrix} 0 & 2 & 1 \\ 2 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$0 + 2x + 2y - 3x - 0 - 4 = 0 \quad \therefore \quad \boxed{x - 2y + 4 = 0}$$

SOLUÇÃO 03

$$y = mx + n$$

$$(i) (0, 2) : 2 = m \cdot 0 + n \quad \therefore \quad \boxed{n = 2}$$

$$(ii) (2, 3) : 3 = m \cdot 2 + 2 \quad \therefore \quad \boxed{m = \frac{1}{2}}$$

$$y = \frac{1}{2}x + 2$$



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$$\boxed{x - 2y + 4 = 0}$$

EXEMPLO 02

Encontre a equação reduzida da reta que passa pelos pontos $(1,2)$ e $(3,5)$.

R :

$$y = m \cdot x + n$$

$$(i) \quad m = \tan \theta = \frac{\Delta y}{\Delta x} = \frac{5 - 2}{3 - 1} = \frac{3}{2}$$

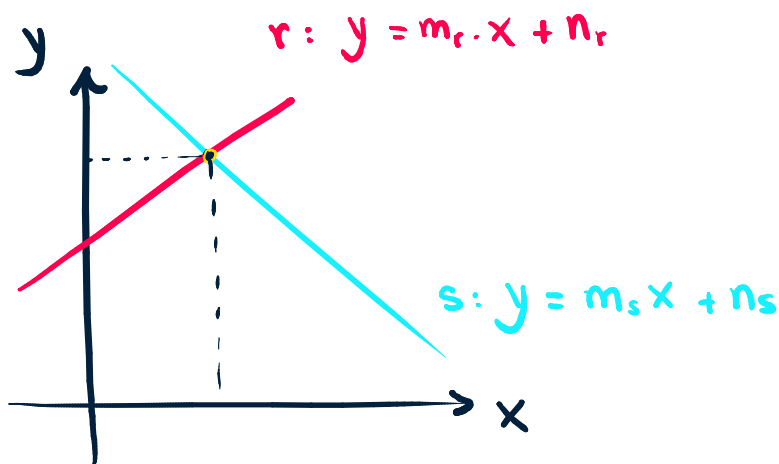
$$(ii) \quad y = \frac{3}{2}x + n$$
$$2 = \frac{3}{2} \cdot 1 + n \quad \leftarrow (1,2)$$

$$n = \frac{1}{2}$$

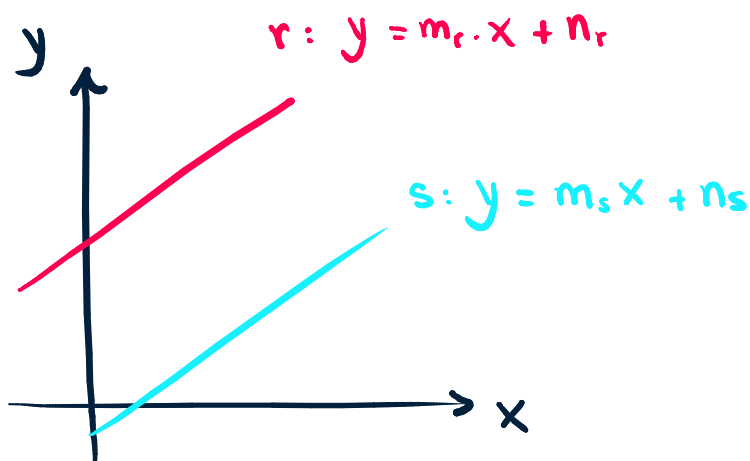
$$y = mx + n \quad \therefore \boxed{y = \frac{3}{2}x + \frac{1}{2}}$$



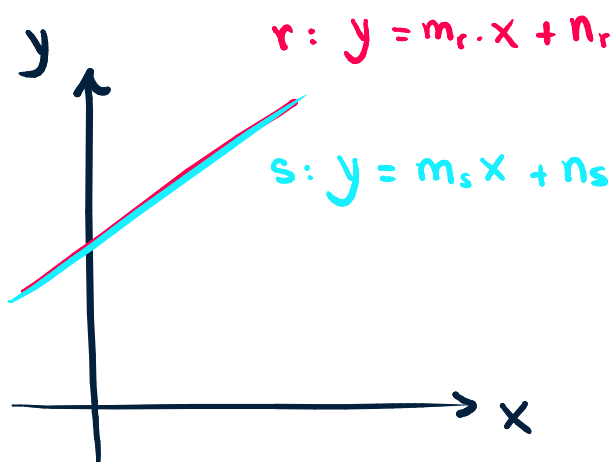
• INTERSECÇÃO DE DUAS RETAS •



- Sistema com solução única: retas concorrentes



- Sistema sem solução: retas paralelas: $m_r = m_s$
 $n_r \neq n_s$



- Sistema com infinitas soluções: retas coincidentes:

$$m_r = m_s$$

$$n_r = n_s$$



EXEMPLO

Encontre o ponto de intersecção da reta

$r: y = 2x - 1$ com a reta:

a) $y = -\frac{1}{2}x + 3$

b) $y - 2x = 4$

c) $2y = 4x - 2$

R :

a)
$$\begin{cases} y = 2x - 1 \\ y = -\frac{1}{2}x + 3 \end{cases} \xrightarrow{\times 4} \begin{cases} y = 2x - 1 \\ 4y = -2x + 12 \end{cases} \quad (+)$$

$$5y = 11 \therefore y = 11/2$$

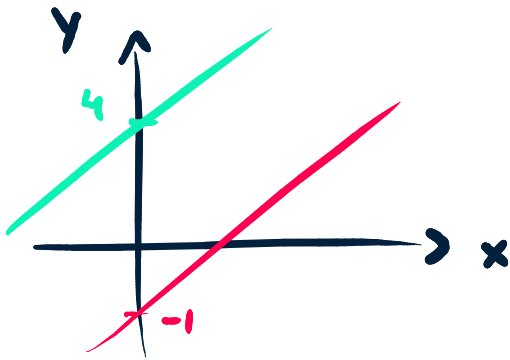
$\rightarrow y = 11/2 ;$

$$11/2 = -\frac{1}{2}x + 3 \therefore 11 = -x + 6 \therefore x = -5$$

$$P = (-5, 11/2)$$



$$b) \begin{cases} y = 2x - 1 \\ y - 2x = 4 \end{cases} \rightarrow \begin{cases} y = 2x - 1 \quad (*) \\ y = 2x + 4 \quad (†) \end{cases}$$



Sistema sem solução!

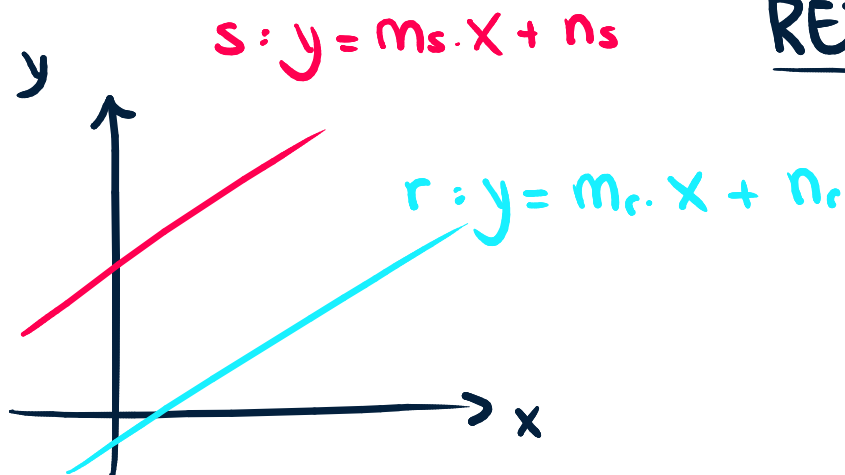
$$c) \begin{cases} y = 2x - 1 \\ 2y = 4x - 2 \end{cases} \xrightarrow{\div 2} \begin{cases} y = 2x - 1 \\ y = 2x - 1 \end{cases}$$

R: TODOS (SÃO A MESMA RETA)



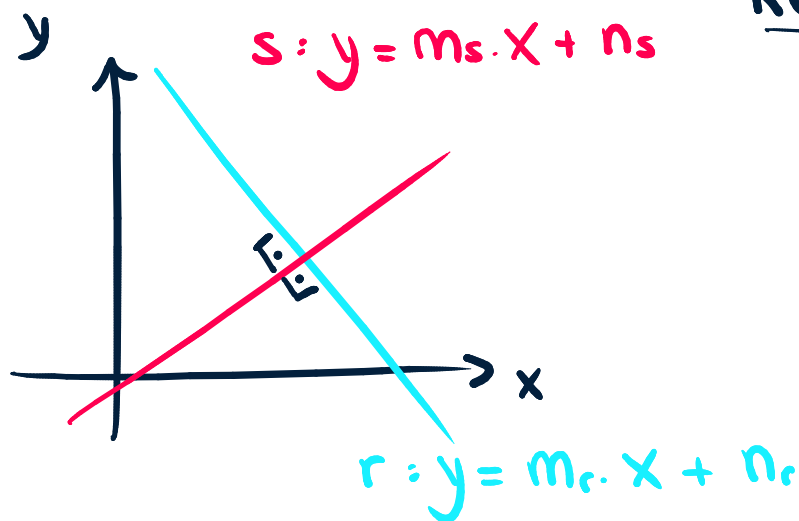
• PARALELISMO e PERPENDICULARISMO •

RETAS PARALELAS



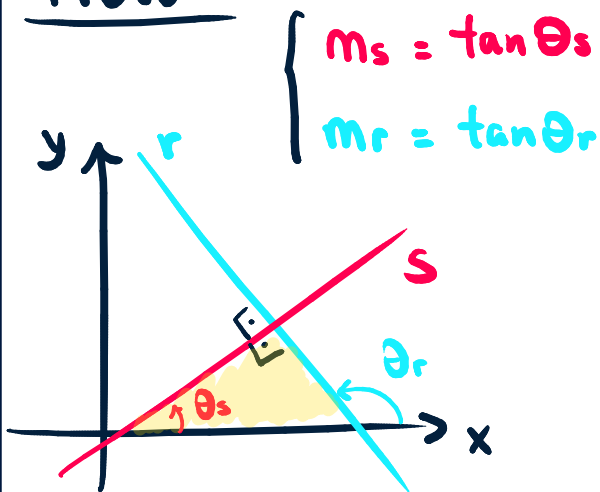
$$m_r = m_s$$

RETAS PERPENDICULARES



$$m_r \cdot m_s = -1$$

Prova:



$$\theta_r = 90^\circ + \theta_s$$

$$\theta_r - \theta_s = 90^\circ$$

$$\tan(\theta_r - \theta_s) = \tan 90^\circ$$



$$\tan(\theta_r - \theta_s) = \tan 90^\circ$$

$$\tan(\theta_r - \theta_s) = \frac{\tan \theta_r - \tan \theta_s}{1 + \tan \theta_r \cdot \tan \theta_s} = \tan 90^\circ$$

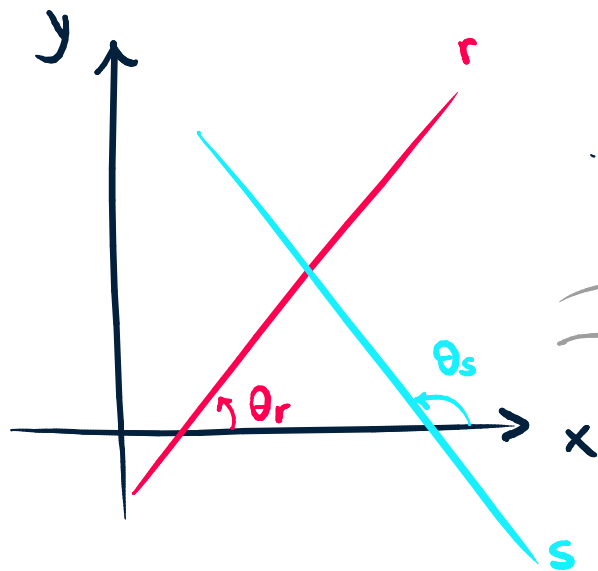
$$\rightarrow 1 + \tan \theta_r \cdot \tan \theta_s = 0 \quad \left\{ \begin{array}{l} m_s = \tan \theta_s \\ m_r = \tan \theta_r \end{array} \right.$$

$$1 + m_r \cdot m_s = 0$$

$$m_r \cdot m_s = -1 \quad \square$$



• ÂNGULO ENTRE RETAS •

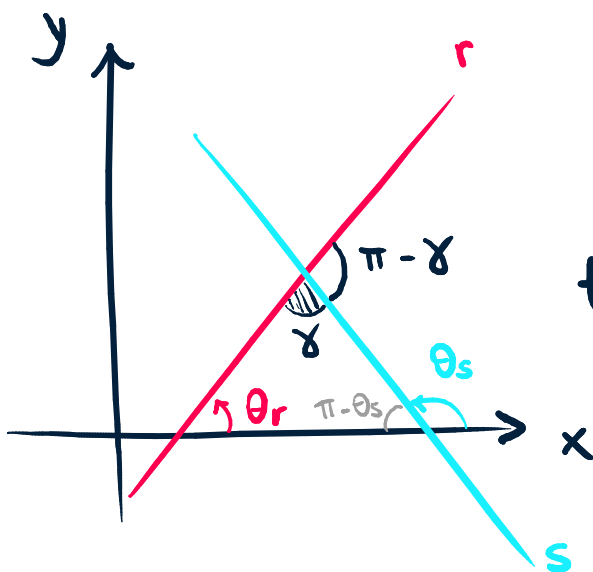


$$r : y = m_r \cdot x + n_r$$

$$s : y = m_s \cdot x + n_s$$

$$m_r = \tan \theta_r$$

$$m_s = \tan \theta_s$$



$$\gamma + \theta_r + \pi - \theta_s = \pi$$

$$\gamma = \theta_s - \theta_r$$

$$\tan \gamma = \tan(\theta_s - \theta_r) = \frac{\tan \theta_s - \tan \theta_r}{1 + \tan \theta_s \cdot \tan \theta_r}$$

$$\tan \gamma = \frac{m_s - m_r}{1 + m_s \cdot m_r}$$

$$\tan \gamma = \pm \frac{m_s - m_r}{1 + m_r \cdot m_s}$$

$$\tan(\pi - \gamma) = -\tan \gamma$$



EXEMPLO

Dada a reta $r: y = 3x - 2$ e

$s: x + y + 1 = 0$, calcule o ângulo entre elas.

R :

$$r: y = 3x - 2 \quad \therefore m_r = 3$$

$$s: y = -x - 1 \quad \therefore m_s = -1$$

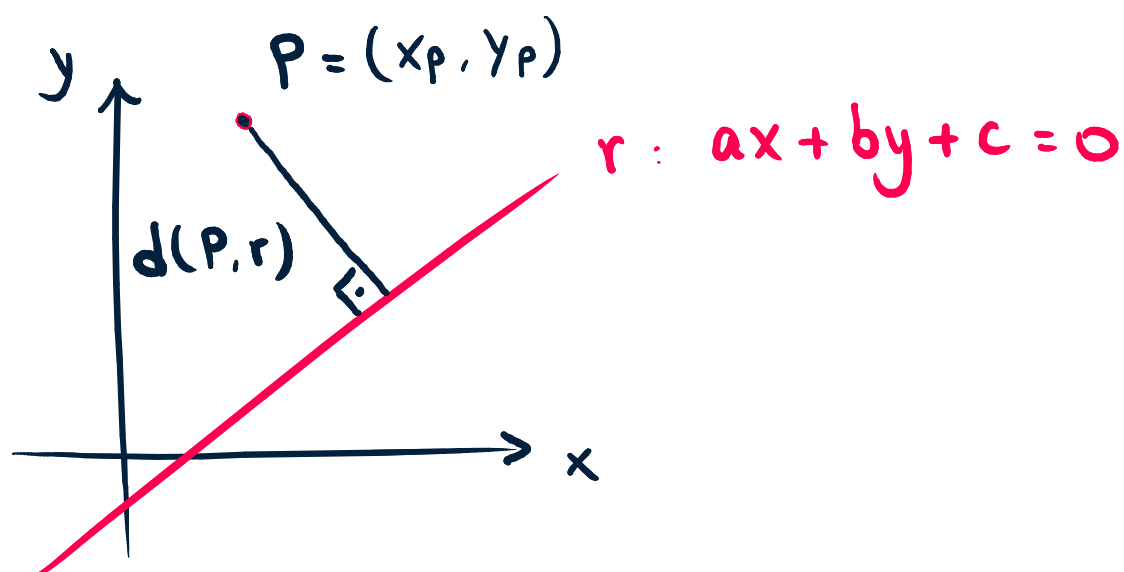
$$\tan \gamma = \pm \frac{m_s - m_r}{1 + m_r \cdot m_s} = \pm \frac{-1 - 3}{1 + 3 \cdot (-1)} = \pm \frac{-4}{-2}$$

$$\tan \gamma = \pm 2$$

$$\gamma = \arctan(2) \text{ ou } \arctan(-2)$$



• DISTÂNCIA : PONTO e RETA •



$$d = \frac{|ax_p + by_p + c|}{\sqrt{a^2 + b^2}}$$



EXEMPLO

Qual é a distância do ponto $P = (1, 2)$ à reta $r: y = 3x + 1$?

R:

$$y = 3x + 1 \therefore 3x - y + 1 = 0$$

$$a = 3 \therefore b = -1 \therefore c = 1$$

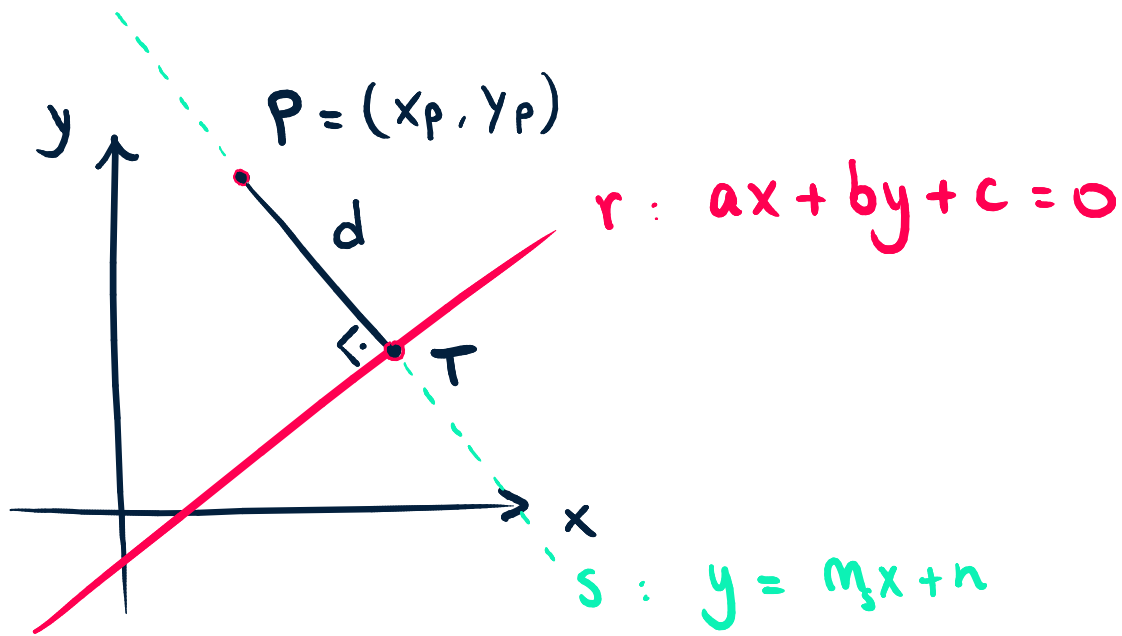
$$d(P, r) = \frac{|ax_p + by_p + c|}{\sqrt{a^2 + b^2}} = \frac{|3x_p - 1y_p + 1|}{\sqrt{3^2 + (-1)^2}}$$

$$= \frac{|3 \cdot 1 - 1 \cdot 2 + 1|}{\sqrt{10}} = \frac{|2|}{\sqrt{10}} = \frac{2}{\sqrt{10}}$$

$$= \frac{2\sqrt{10}}{10}$$



DEMONSTRAÇÃO



$$r: ax + by + c = 0 \quad \therefore by = -ax - c$$

$$y = -\frac{a}{b}x - \frac{c}{b}$$

$$(i) \quad m_s \cdot m_r = -1 \quad \therefore m_s \left(-\frac{a}{b} \right) = -1 \quad \therefore m_s = \frac{b}{a}$$

$$(ii) \quad y = \frac{b}{a}x + n$$

O ponto $P = (x_p, y_p) \in s$:

$$y_p = \frac{b}{a}x_p + n \quad \therefore n = \frac{a y_p}{a} - \frac{b x_p}{a}$$

$$s: y = \frac{b}{a}x + \frac{(a \cdot y_p - b x_p)}{a}$$

(iii) $s \cap r$:

$$-\frac{a}{b}x - \frac{c}{b} = \frac{b}{a}x + \frac{(a \cdot y_p - b x_p)}{a} \quad \xrightarrow{\times (ab)}$$

$$-a^2x - ac = b^2x + b(a y_p - b x_p)$$

$$x_T = \frac{b(b x_p - a y_p) - ac}{(a^2 + b^2)}$$

Pela reta r : $y_T = -\frac{a}{b}x_T - \frac{c}{b}$

$$y_T = \frac{a}{b} \cdot \frac{b(a y_p - b x_p)}{(a^2 + b^2)} + \frac{a}{b} \cdot \frac{ac}{(a^2 + b^2)} - \frac{c}{b}$$

$$y_T = \frac{a(a y_p - b x_p) + a^2 c/b - c(a^2 + b^2)/b}{(a^2 + b^2)}$$

(iv) $d(P, T) = \sqrt{\Delta x^2 + \Delta y^2}$

$$= \sqrt{(x_T - x_p)^2 + (y_T - y_p)^2}$$



$$\rightarrow (x_T - x_P)^2 = \left[\frac{b(bx_P - ay_P) - ac}{(a^2 + b^2)} - x_P \right]^2$$

$$= \left[\frac{\cancel{b^2}x_P - aby_P - ac - x_P \cdot \cancel{a^2} - \cancel{x_P b^2}}{(a^2 + b^2)} \right]^2$$

$$= \frac{(aby_P + ac + x_P \cdot a^2)^2}{(a^2 + b^2)^2}$$

$$\rightarrow (y_T - y_P)^2 =$$

$$= \frac{a(\cancel{a}y_P - bx_P) + a^2c/b - c \cdot (a^2 + b^2)/b - \frac{y_P(\cancel{a^2 + b^2})}{(a^2 + b^2)}}{(a^2 + b^2)}$$

$$= \frac{(-abx_P - b^2y_P + a^2c/b - c \cdot (a^2 + b^2)/b)^2}{(a^2 + b^2)^2}$$



$$d^2(P, T) = \Delta x^2 + \Delta y^2 =$$

$$= \frac{(aby_p + ac + x_p \cdot a^2)^2 + (-abx_p - b^2y_p + \cancel{a^2c/b} - \cancel{ca^2/b} - cb)^2}{(a^2 + b^2)^2}$$

$$= \frac{a^2 \cdot (by_p + c + x_p \cdot a)^2 + b^2 (a \cdot x_p + by_p + c)^2}{(a^2 + b^2)^2}$$

$$= \frac{(a \cdot x_p + by_p + c)^2 \cancel{(a^2 + b^2)}}{\cancel{(a^2 + b^2)^2}} = \frac{(a \cdot x_p + by_p + c)^2}{(a^2 + b^2)}$$

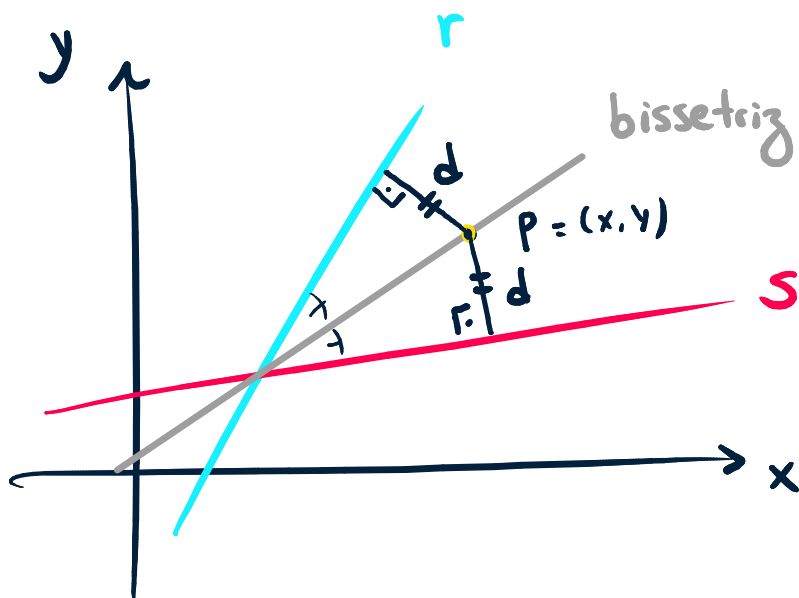
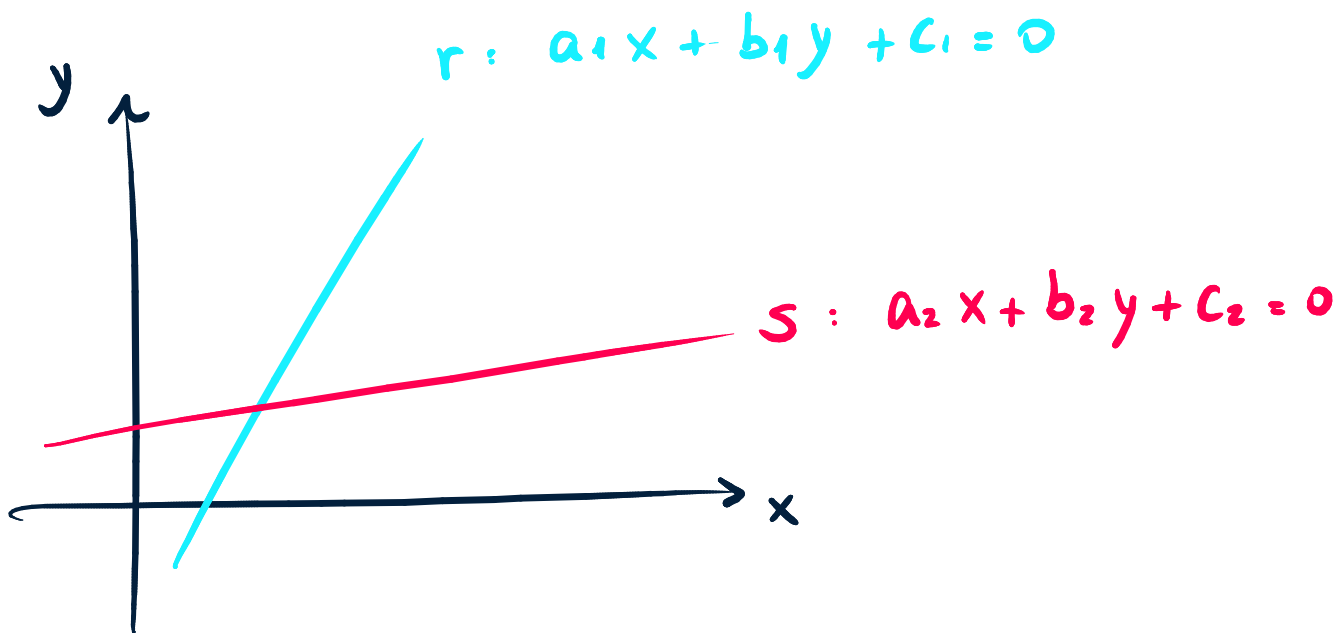
$$d(P, r) = \sqrt{\frac{(a \cdot x_p + by_p + c)^2}{(a^2 + b^2)}}$$

$$d(P, r) = \frac{|a \cdot x_p + by_p + c|}{\sqrt{a^2 + b^2}}$$

□



• BISSETRIZ DE DUAS RETAS •



$$d(P, r) = d(P, s)$$

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$



EXEMPLO

Dada a reta $r: 2x - y + 3 = 0$ e a reta $s: x + y = 0$, obtenha as bissetrizes dos ângulos formados pelas retas.

$$\underline{R}: \frac{2x - 1 \cdot y + 3}{\sqrt{2^2 + (-1)^2}} = \pm \frac{1 \cdot x + 1 \cdot y + 0}{\sqrt{1^2 + 1^2}}$$
$$\frac{2x - 1 \cdot y + 3}{\sqrt{5}} = \pm \frac{1 \cdot x + 1 \cdot y + 0}{\sqrt{2}}$$

$$(i) 2\sqrt{2}x - \sqrt{2}y + 3\sqrt{2} = x\sqrt{5} + y\sqrt{5}$$

$$y(\sqrt{5} + \sqrt{2}) = x(2\sqrt{2} - \sqrt{5}) + 3\sqrt{2}$$

$$y = \left(\frac{2\sqrt{2} - \sqrt{5}}{\sqrt{5} + \sqrt{2}} \right) x + \frac{3\sqrt{2}}{(\sqrt{5} + \sqrt{2})}$$

$$(ii) 2\sqrt{2}x - \sqrt{2}y + 3\sqrt{2} = -x\sqrt{5} - y\sqrt{5}$$

$$y = \left(\frac{-\sqrt{5} - 2\sqrt{2}}{\sqrt{5} - \sqrt{2}} \right) x - \frac{3\sqrt{2}}{(\sqrt{5} - \sqrt{2})}$$



• ÁREA DE POLÍGONOS •

Dado um triângulo de vértices $A = (x_A, y_A)$,
 $B = (x_B, y_B)$ e $C = (x_C, y_C)$, sua área é
dada por

$$S_{\triangle ABC} = \frac{1}{2} |D_{ABC}|$$

em que

$$D_{ABC} = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

ou, se preferir:

$$D_{ABC} = D_1 - D_2, \text{ onde}$$



$$D_1 = \begin{vmatrix} x_A & y_A \\ x_B & y_B \\ x_C & y_C \\ x_A & y_A \end{vmatrix} = x_A y_B + x_B y_C + x_C y_A$$

$$D_2 = \begin{vmatrix} x_A & y_A \\ x_B & y_B \\ x_C & y_C \\ x_A & y_A \end{vmatrix} = y_A x_B + y_B x_C + y_C x_A$$



EXEMPLO

Seja $(0,0)$, $(2,4)$ e $(5, 5/2)$ os vértices de um Δ . Calcule sua área.

R :

$$i) D_{ABC} = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 4 & 1 \\ 5 & 5/2 & 1 \end{vmatrix}$$

$$D_{ABC} = 5 - 20 = -15$$

$$S_{ABC} = \frac{1}{2} |D_{ABC}| = \frac{15}{2}$$

$$(ii) D_{ABC} = \overset{5}{D_1} - \overset{20}{D_2} = 5 - 20 = -15$$

$$\begin{vmatrix} x_A & y_A \\ x_B & y_B \\ x_C & y_C \\ x_A & y_A \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 2 & 4 \\ 5 & 5/2 \\ 0 & 0 \end{vmatrix}$$

$$S_{ABC} = \frac{1}{2} |D_{ABC}|$$
$$= \frac{15}{2}$$



Obs. :: para um polígono qualquer de

vértices $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, ..., $A_n = (x_n, y_n)$,

sua área será :

$$S = \frac{1}{2} |D|$$

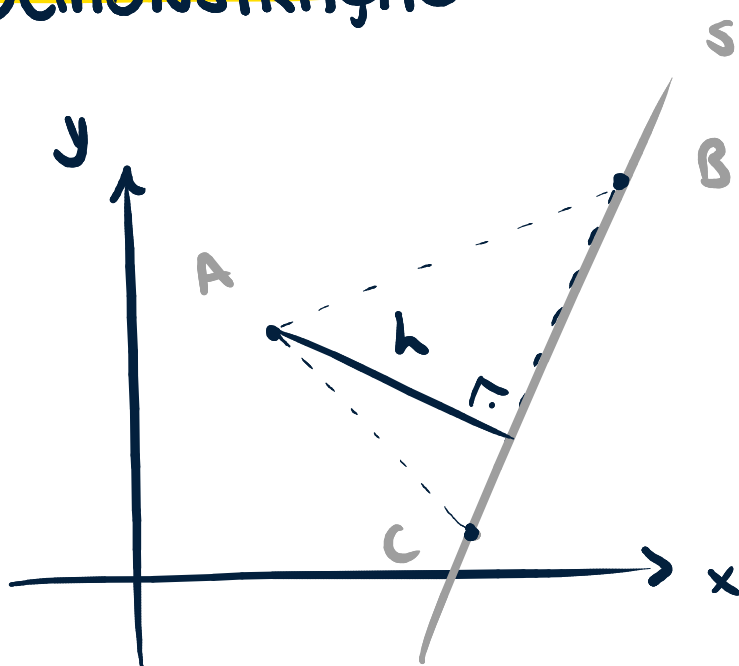
em que $D = D_1 - D_2$:

$$D_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix} =$$

$$D_2 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix} =$$



DEMONSTRAÇÃO



$$A = (x_A, y_A)$$

$$B = (x_B, y_B)$$

$$C = (x_C, y_C)$$

$$(i) \quad B = d(B, C) = \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2} \quad (I)$$

$$(ii) \quad \left. \begin{array}{l} \text{reta } s: y = mx + n \\ m = \frac{\Delta y}{\Delta x} = \frac{y_B - y_C}{x_B - x_C} \end{array} \right\} y = \left(\frac{y_B - y_C}{x_B - x_C} \right) x + n$$

Ponto B :

$$y_B = \left(\frac{y_B - y_C}{x_B - x_C} \right) x_B + n \quad \therefore \quad n = y_B - \left(\frac{y_B - y_C}{x_B - x_C} \right) x_B$$

$$y = \frac{(y_B - y_C)}{(x_B - x_C)} x + \frac{y_B (x_B - x_C) - (y_B - y_C) x_B}{(x_B - x_C)}$$



$$ax + by + c = 0$$

$$(y_B - y_C)x - (x_B - x_C)y + y_B(x_B - x_C) - (y_B - y_C)x_B = 0$$

$$(iii) h = d(A, s) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

$$h = \frac{(y_B - y_C)x_A - (x_B - x_C)y_A + y_B(x_B - x_C) - x_B(y_B - y_C)}{\sqrt{(x_B - x_C)^2 + (y_B - y_C)^2}} \quad (\text{III})$$

$$2 \cdot S_{\Delta ABC} = B \cdot h = (y_B - y_C)x_A - (x_B - x_C)y_A + y_B(x_B - x_C) - x_B(y_B - y_C)$$

$$= y_B \cdot x_A - y_C x_A - x_B \cdot y_A + x_C y_A + \cancel{y_B x_B} - y_B x_C - \cancel{x_B y_B} + x_B y_C$$

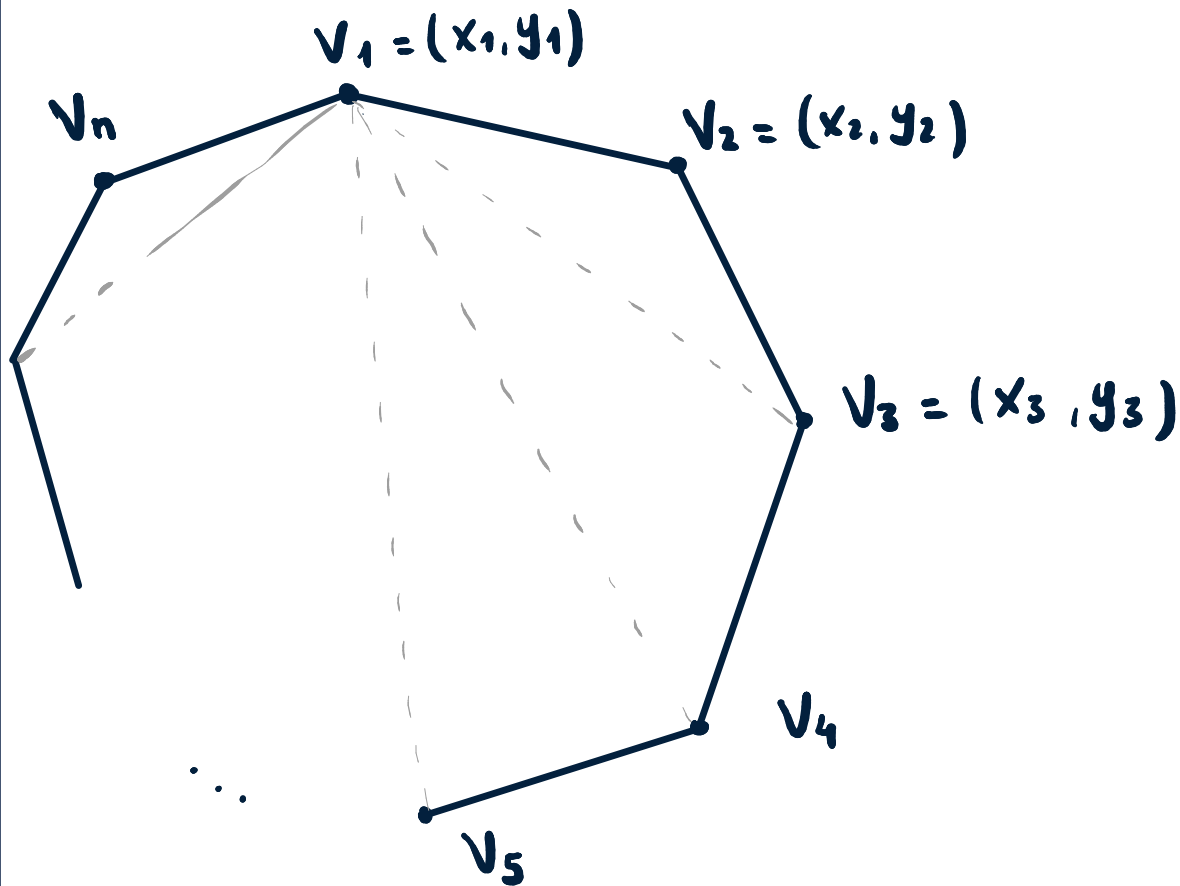
$$= y_B \cdot x_A - y_C x_A - x_B \cdot y_A + x_C y_A - y_B x_C + x_B y_C$$

$$2S_{\Delta ABC} = |D| = D_1 - D_2$$

$$S_{\Delta ABC} = \frac{1}{2} |D|$$



POLIGONO QUALQUER



$$\Delta V_1 V_2 V_3 + \Delta V_1 V_3 V_4 + \dots$$

$$S = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_1 & y_1 \end{vmatrix} \dots$$



$$S = \frac{1}{2} |D|$$

em que $D = D_1 - D_2$:

$$D_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix} =$$

$$D_2 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix} =$$

Obs.: os v rtices devem ser consecutivos nas linhas da matriz (sentido H. ou A.H.).

