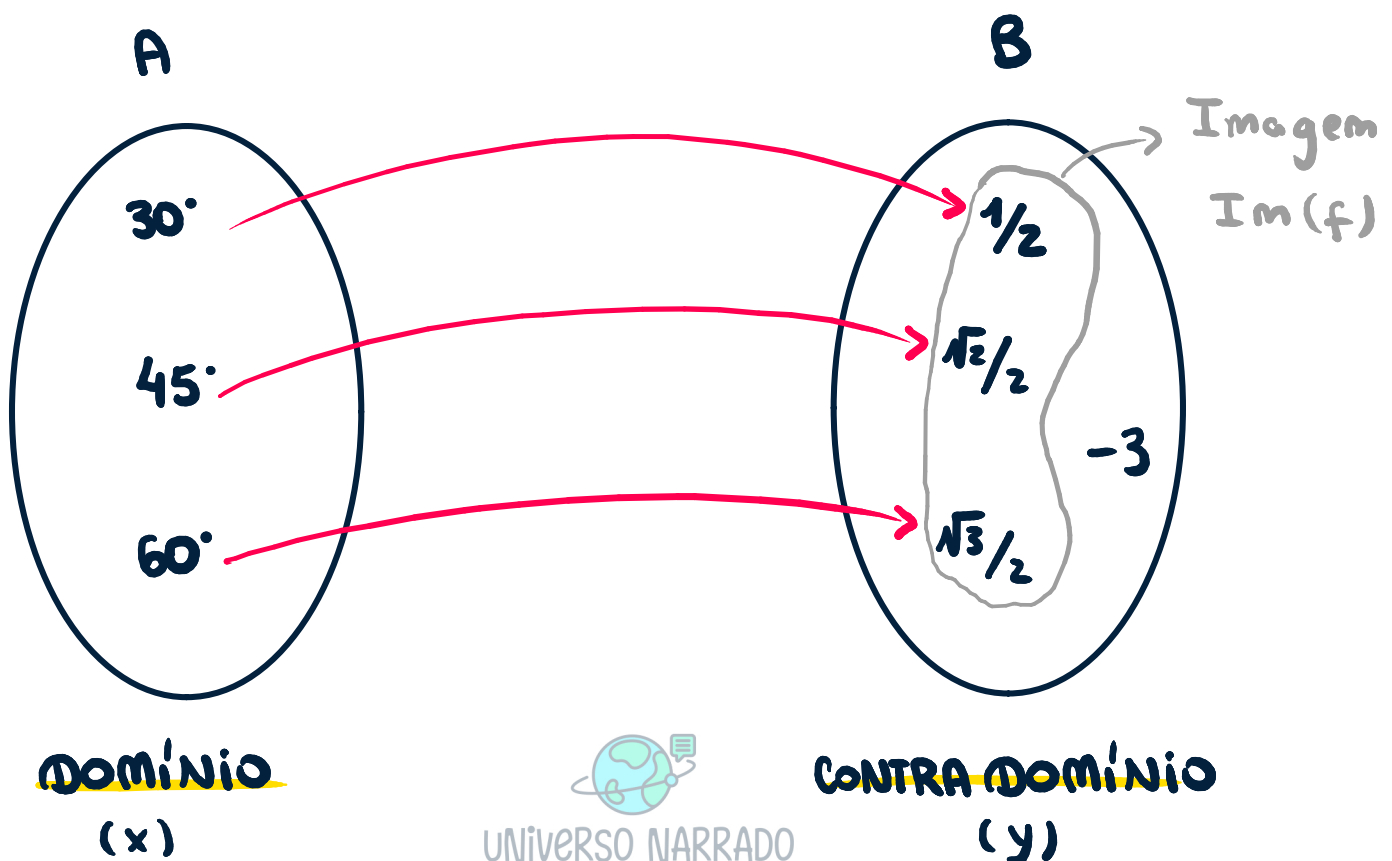


FUNÇÕES TRIGONOMÉTRICAS

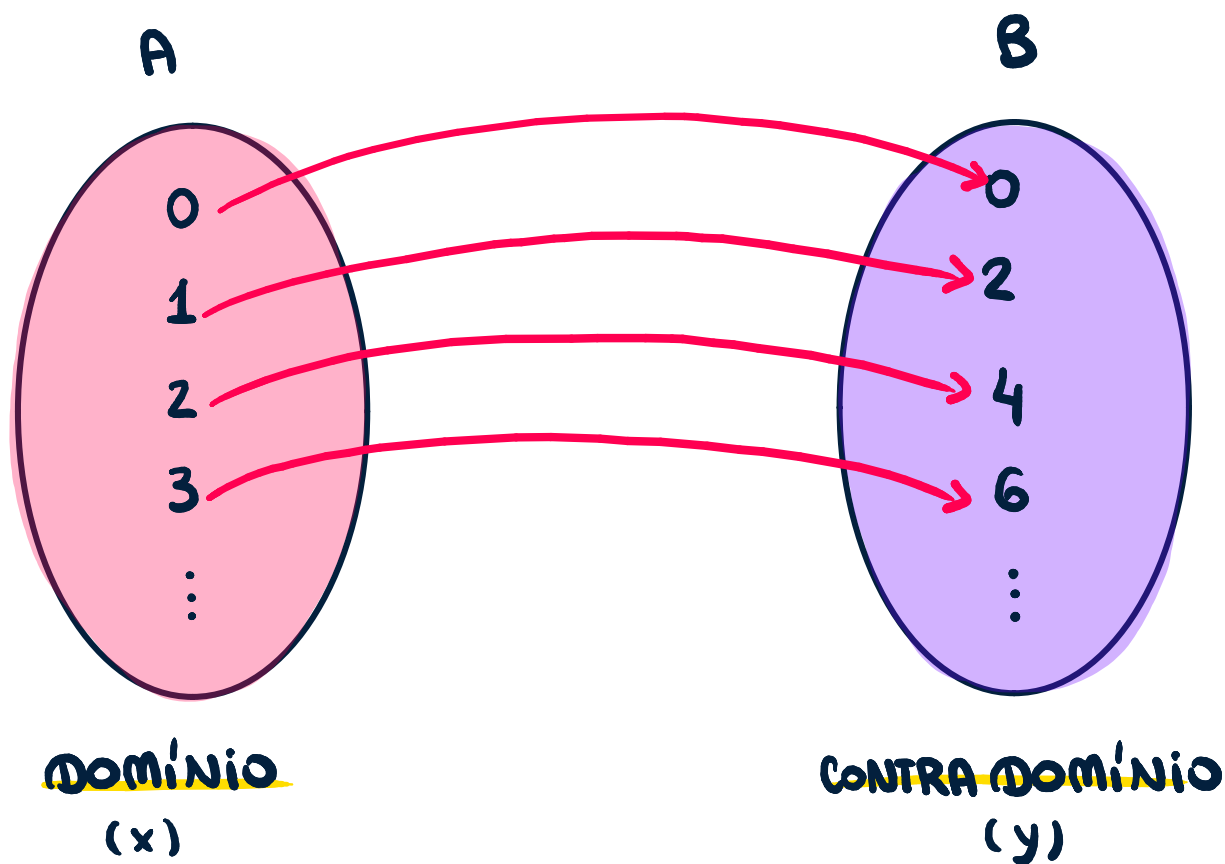
· INTRODUÇÃO ·

- f é uma função de A em B ($f: A \rightarrow B$) se relaciona cada elemento de A a um único elemento de B .



1.1 . VISUALIZANDO UMA FUNÇÃO

a) DIAGRAMA DE FLECHAS



1.1 . VISUALIZANDO UMA FUNÇÃO

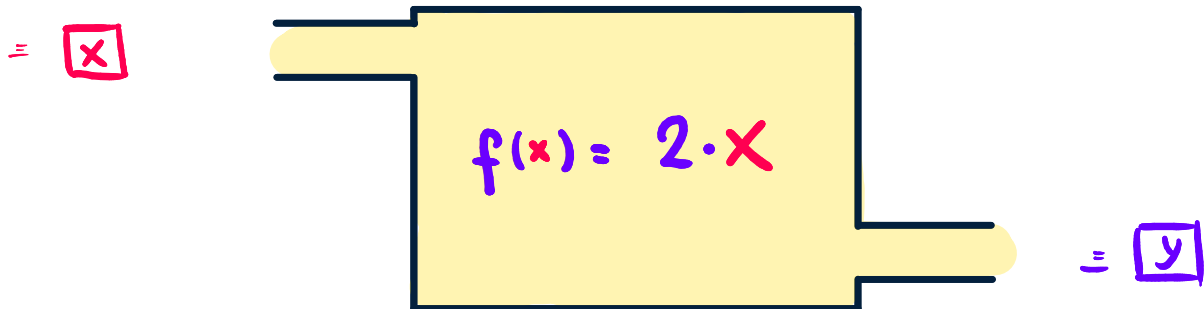
b) TABELA

A	B
0	0
1	2
2	4
3	6
4	8
⋮	



1.1 . VISUALIZANDO UMA FUNÇÃO

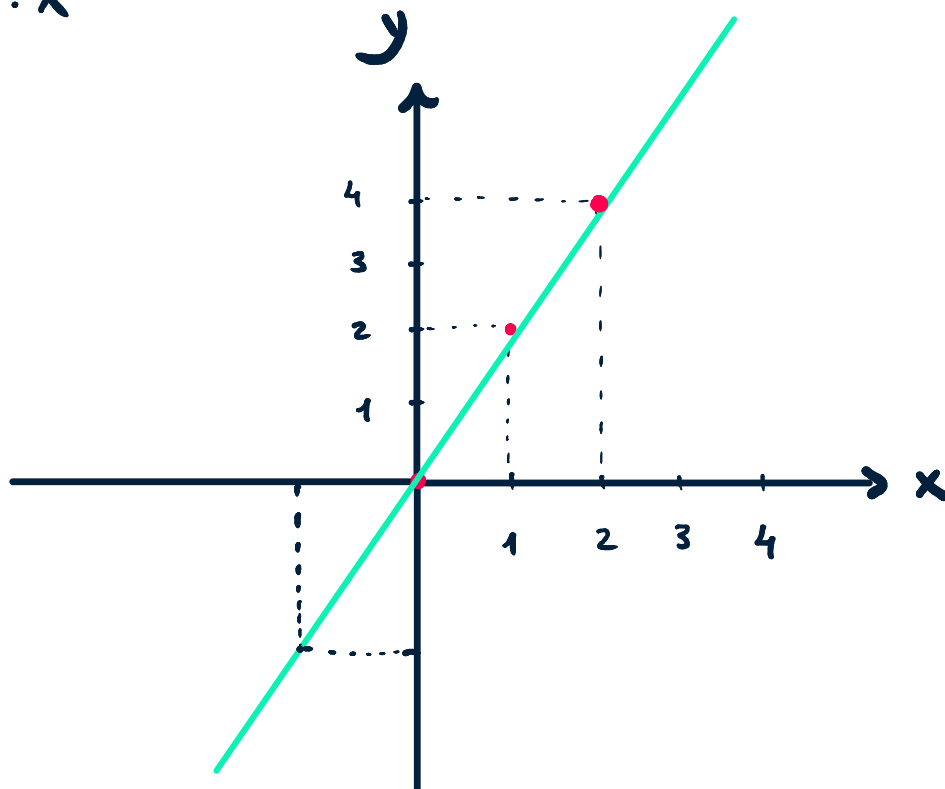
c) MÁQUINA



1.1 . VISUALIZANDO UMA FUNÇÃO

d) GRÁFICO

$$y = 2 \cdot x$$



1.2 • FUNÇÕES PERIÓDICAS

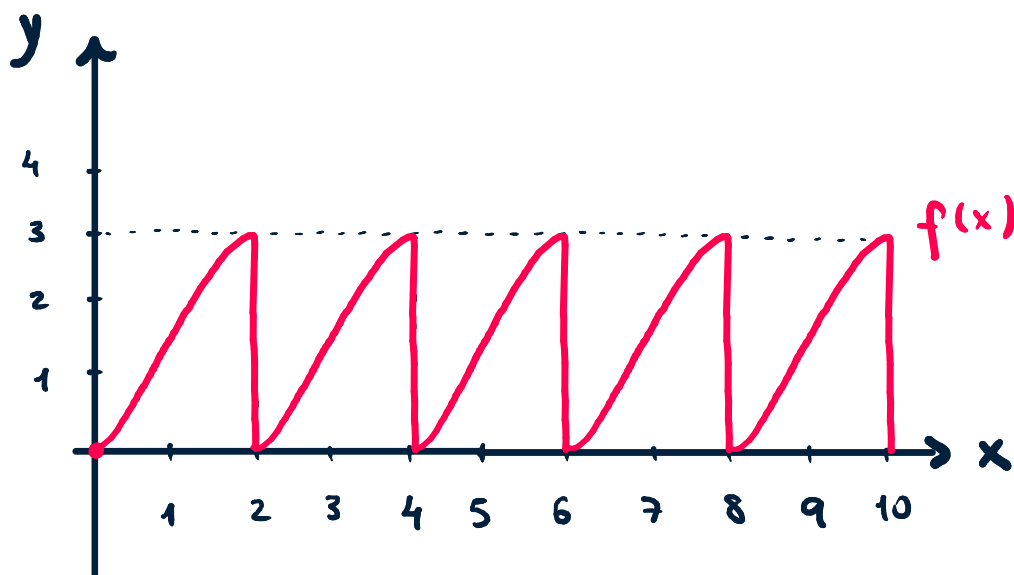
DEFINIÇÃO

Uma função f , de domínio $A \subset \mathbb{R}$, é periódica se existe um n° real T tal que

$$f(x+T) = f(x), \forall x \in A$$

O menor valor positivo de T é o período da função.

EXEMPLO



↙ $f(x+2) = f(x)$

$$\boxed{T = 2}$$



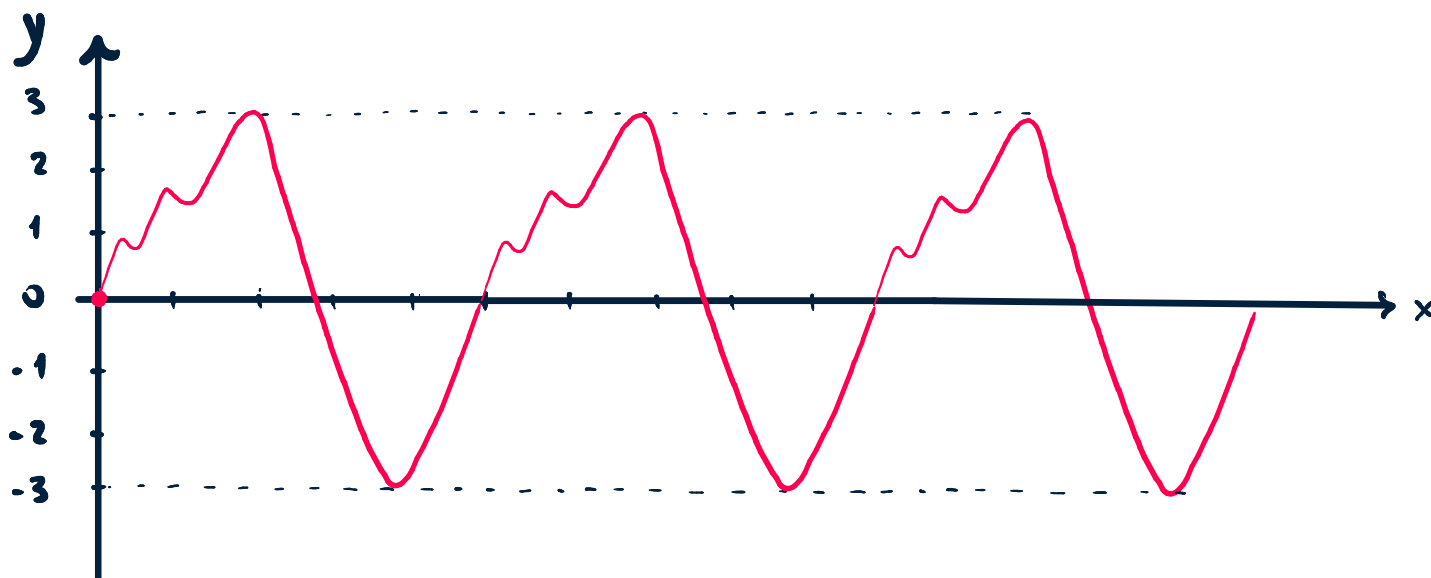
1.3 • FUNÇÕES LIMITADAS

DEFINIÇÃO

Uma função f , de domínio $A \subset \mathbb{R}$, é limitada se existe um n.º real M , positivo, tal que

$$-M \leq f(x) \leq M$$

EXEMPLO



↪ $M = 3$



1.4 • PARIDADE DE FUNÇÕES

DEFINIÇÃO

Uma função f , de domínio $A \subset \mathbb{R}$, tal que se $x \in A$ então $-x \in A$ é dita

.PAR : se $\forall x \in A$ tem-se $f(-x) = f(x)$

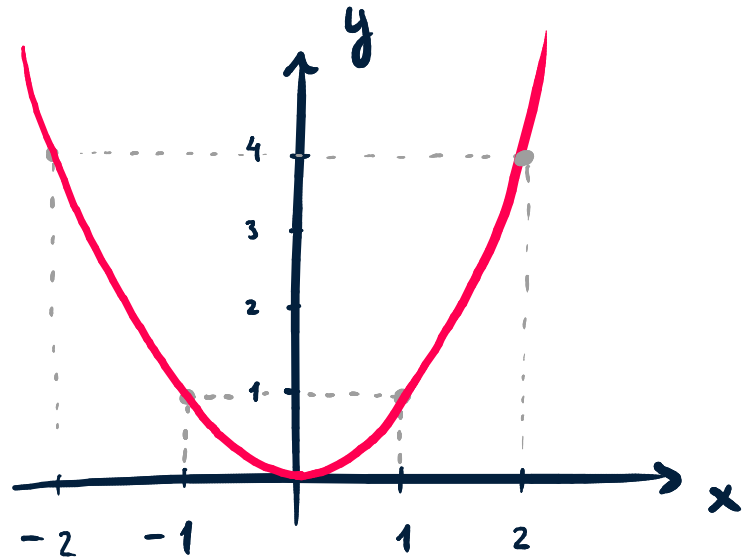
.ÍMPAR : se $\forall x \in A$ tem-se $f(-x) = -f(x)$



EXEMPLO

➡ PAR

(i) $f(x) = x^2$



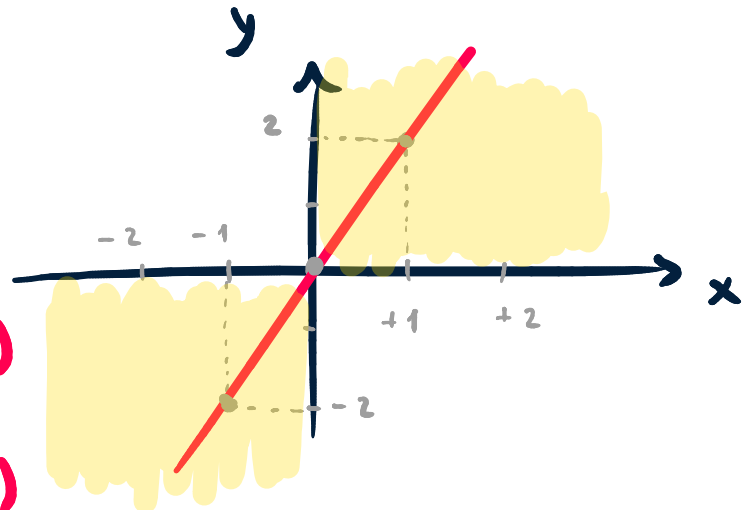
x	f(x)
+1	1
-1	1
+2	4
-2	4

} $f(-1) = f(1)$

} $f(-2) = f(2)$

➡ ÍMPAR

(ii) $f(x) = 2x$



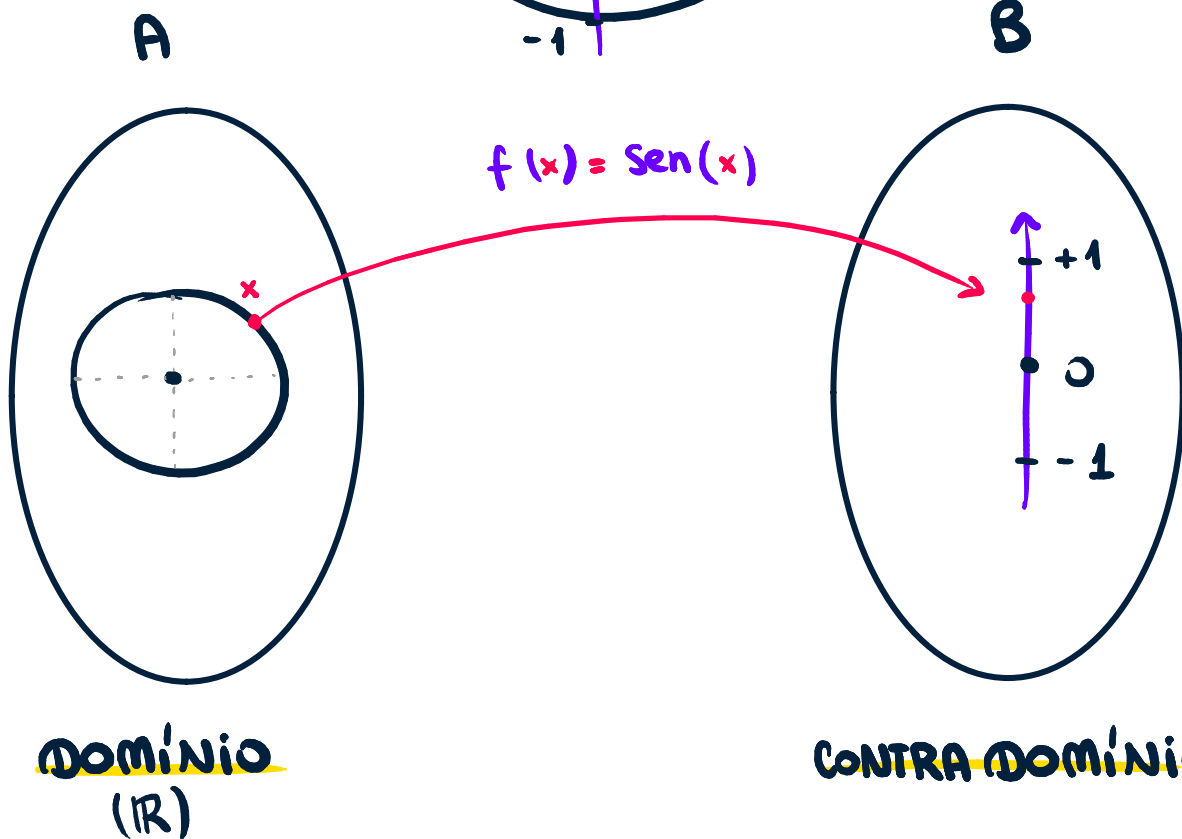
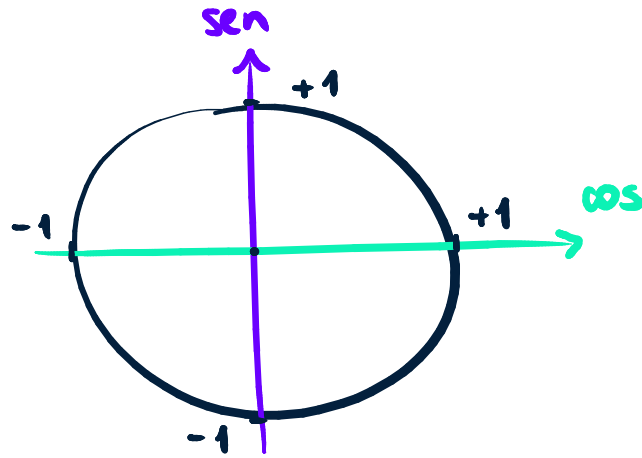
x	f(x)
+1	2
-1	-2
+2	4
-2	-4

} $f(-1) = -f(1)$

} $f(-2) = -f(2)$



· FUNÇÃO SENO ·



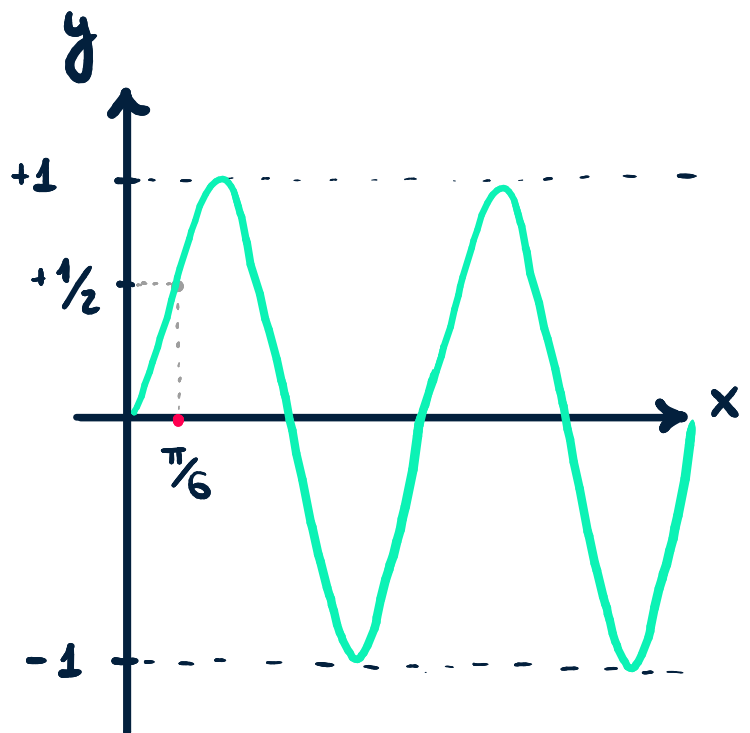
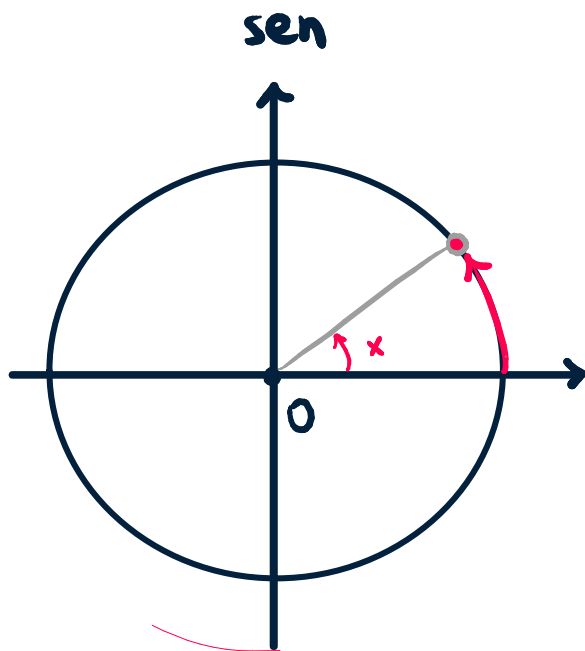
= x

$$f(x) = \text{sen}(x)$$

= y

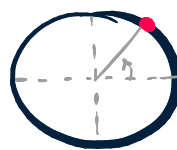


FUNÇÃO SENO: GRÁFICO

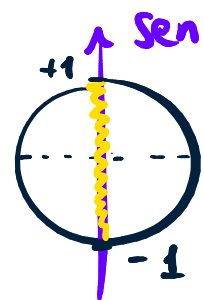


$$f(x) = \text{sen } x$$

• DOMÍNIO : $x \in \mathbb{R}$



• IMAGEM : $y \in \mathbb{R} \mid y \in [-1, 1]$



• PERÍODO : 2π : $\text{sen}(x + 2\pi) = \text{sen } x$

• PARIDADE: ÍMPAR



EXEMPLO 01

Esboce o gráfico de $g(x) = 3\text{sen}(x)$

R :

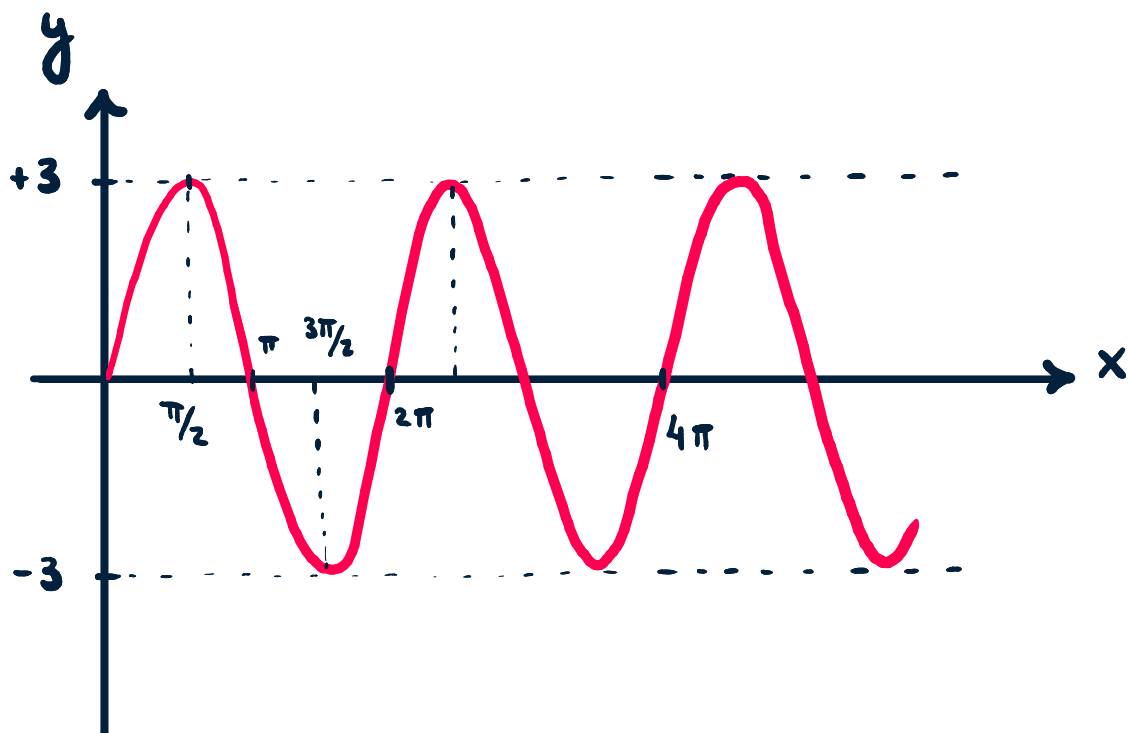
$$-1 \leq \text{sen}(x) \leq +1$$

$$-3 \leq 3\text{sen}(x) \leq +3$$

$\times 3$

$$-3 \leq g(x) \leq +3$$

$$\Rightarrow \text{Im}(g) = [-3, 3]$$



EXEMPLO 02

Esboce o gráfico de $h(x) = 3\text{sen}(x) + 3$

R :

$$-1 \leq \text{sen } x \leq +1$$

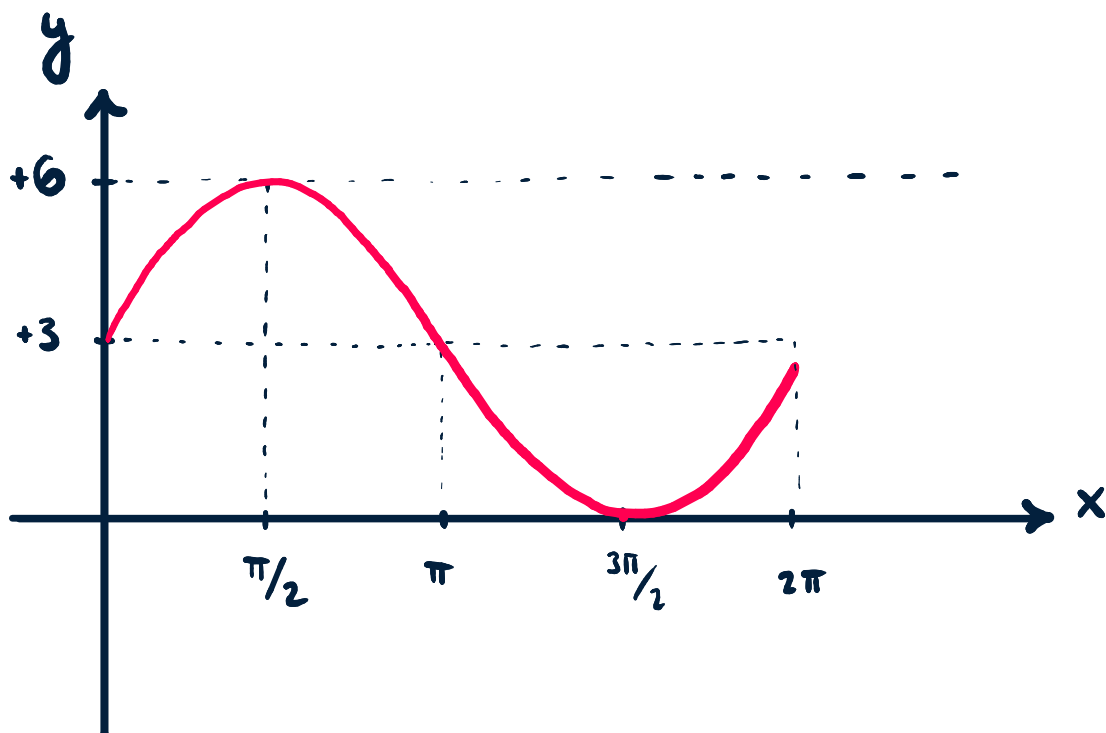
$$-3 \leq 3\text{sen } x \leq +3$$

$$0 \leq h(x) \leq +6$$

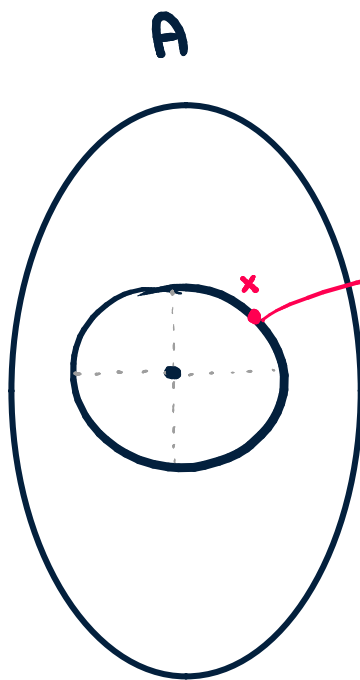
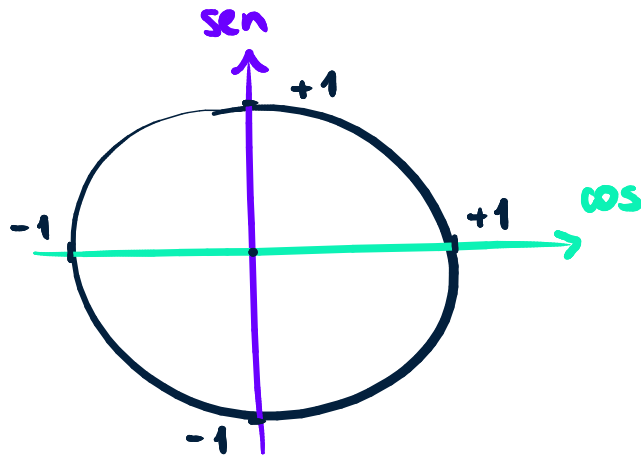
$\times 3$

$+3$

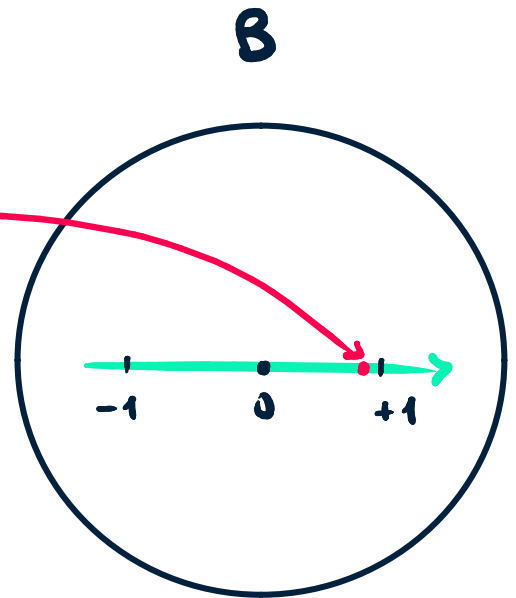
$\hookrightarrow \text{Im}(h) = [0, 6]$



· FUNÇÃO COSSENO ·



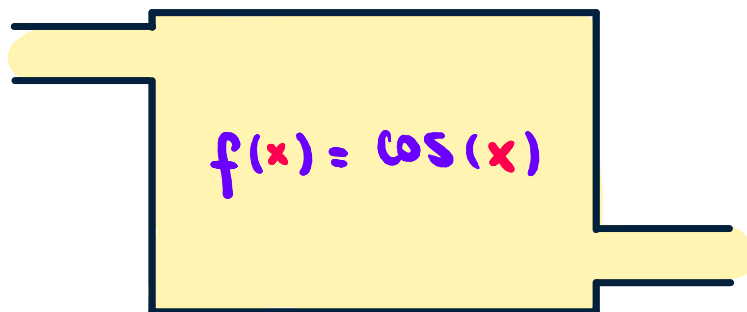
$$f(x) = \cos(x)$$



DOMÍNIO
($\mathbb{R}$)

CONTRA DOMÍNIO

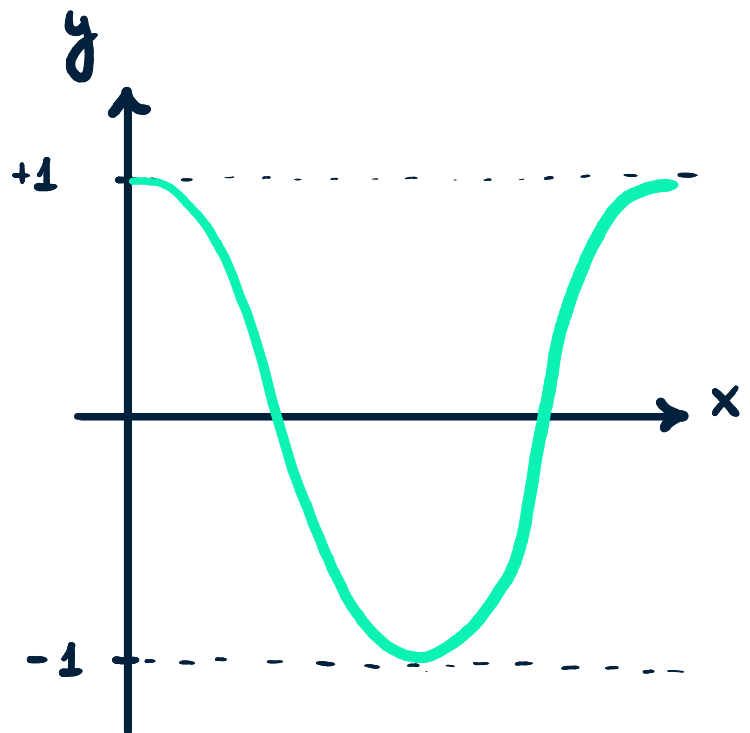
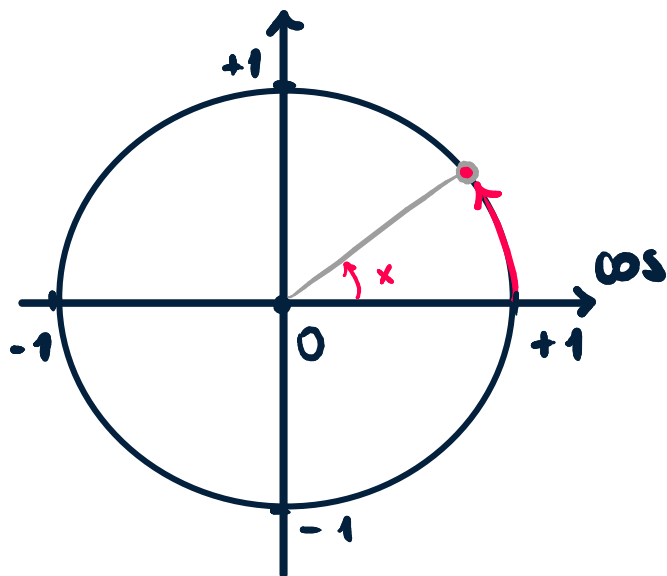
=" x



= y

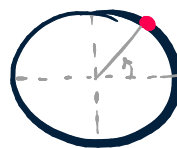


FUNÇÃO COSSENO: GRÁFICO

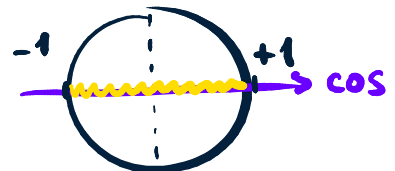


$$f(x) = \cos x$$

• DOMÍNIO : $x \in \mathbb{R}$



• IMAGEM : $y \in \mathbb{R} \mid y \in [-1, 1]$



• PERÍODO : 2π : $\cos(x + 2\pi) = \cos x$

• PARIDADE: PAR



EXEMPLO 01

Esboce o gráfico de $g(x) = \frac{1}{2} \cos x$

R :

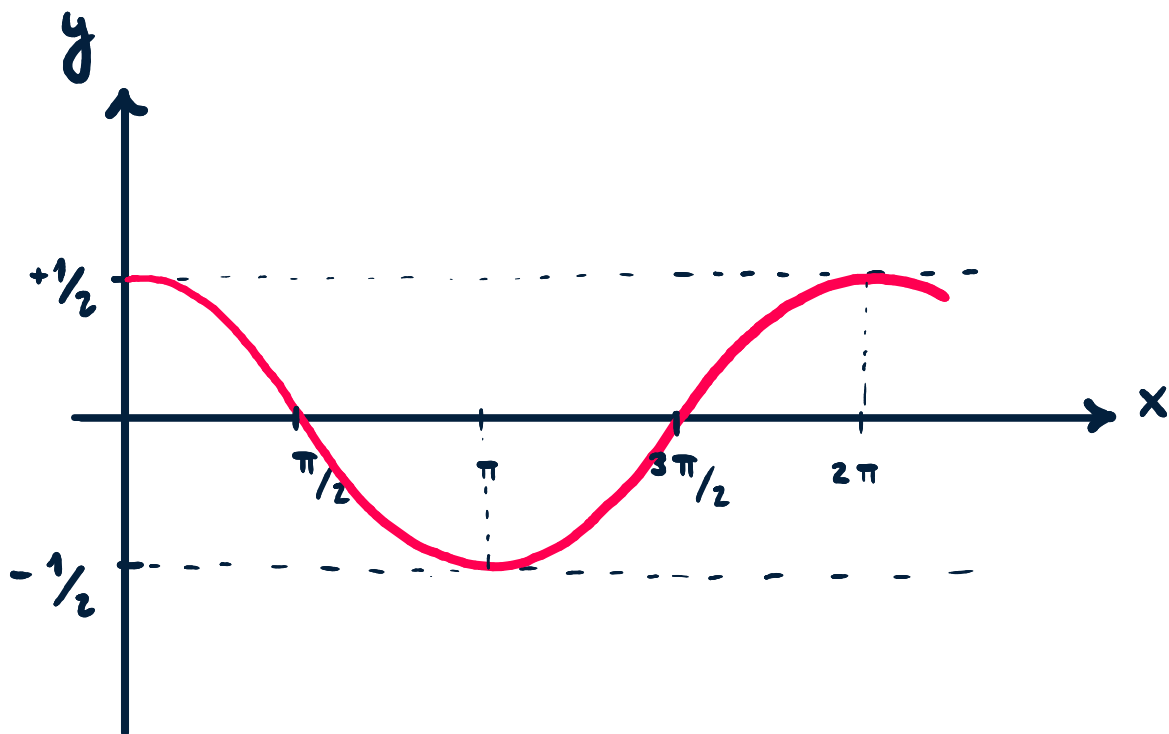
$$-1 \leq \cos x \leq +1$$

$$-\frac{1}{2} \leq g(x) \leq +\frac{1}{2}$$

) $\times \frac{1}{2}$



$$\text{Im}(g) = [-\frac{1}{2}, +\frac{1}{2}]$$



EXEMPLO 02

Esboce o gráfico de $h(x) = \frac{1}{2} \cos x + \frac{3}{2}$

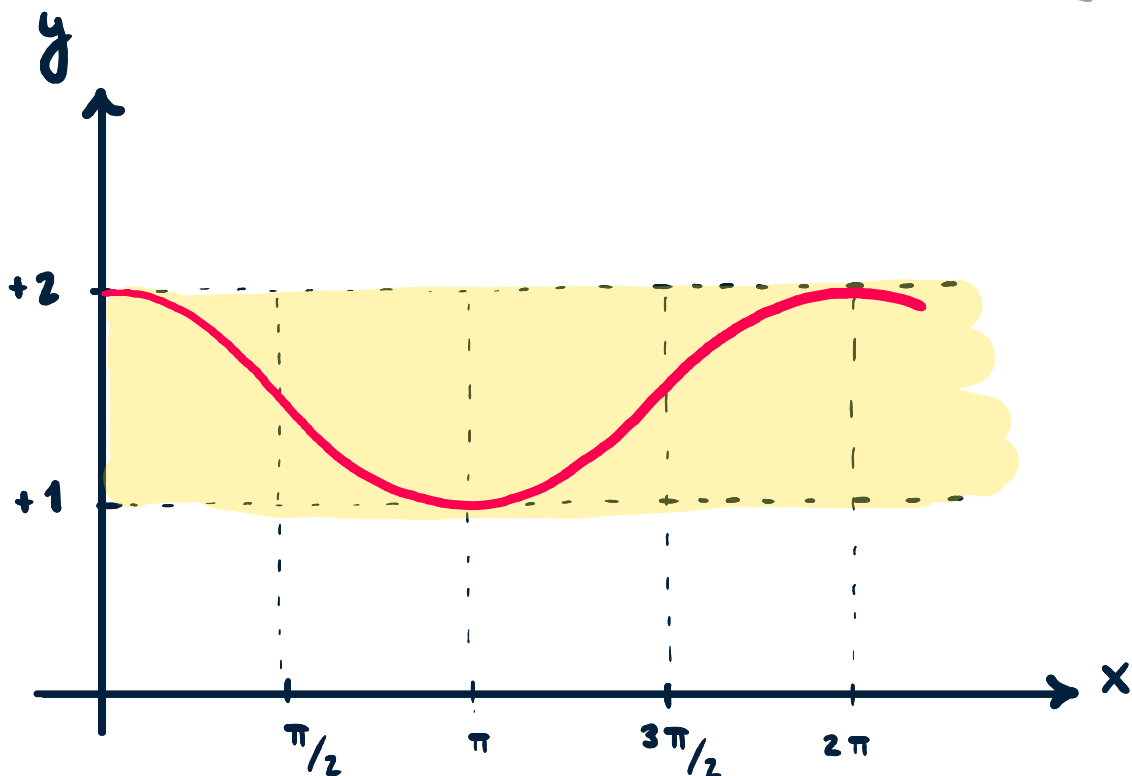
R : $-1 \leq \cos x \leq +1$

$$-\frac{1}{2} \leq \frac{1}{2} \cos x \leq +\frac{1}{2}$$

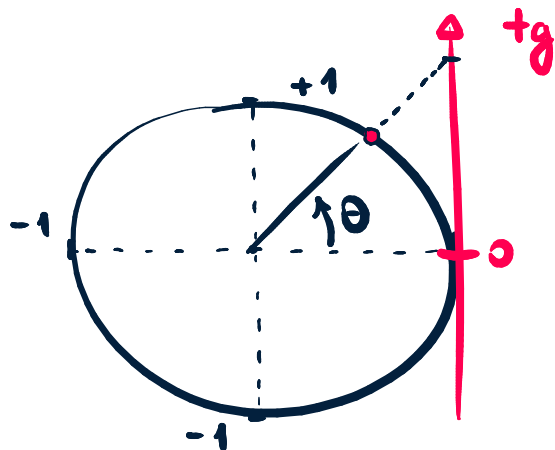
$$+1 \leq \frac{1}{2} \cos x + \frac{3}{2} \leq +2$$

$$+1 \leq h(x) \leq +2$$

$\hookrightarrow \text{Im}(h) = [1, 2]$



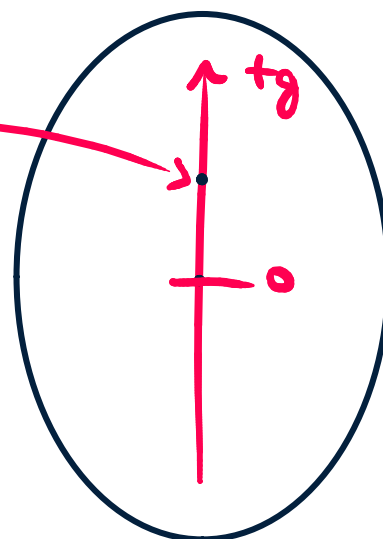
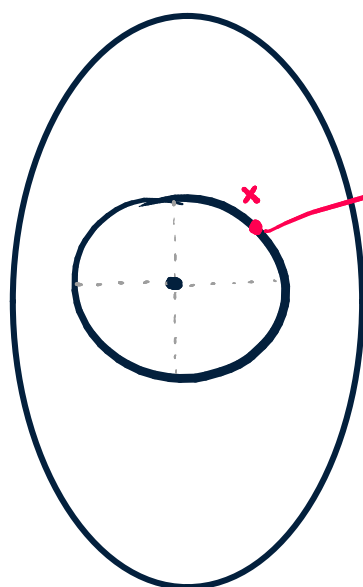
· FUNÇÃO TANGENTE ·



$A(\mathbb{R})$

$B(\mathbb{R})$

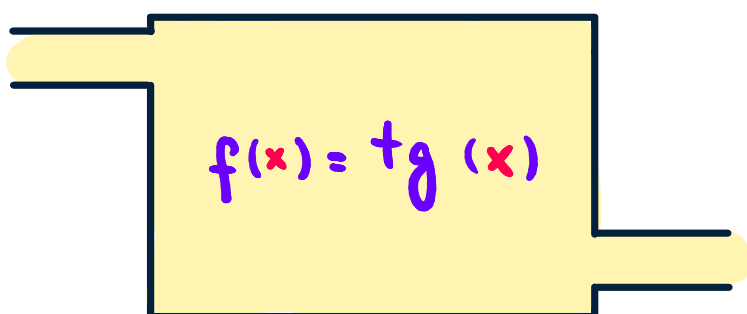
$$f(x) = tg(x)$$



DOMÍNIO

CONTRA DOMÍNIO

$= \boxed{x}$

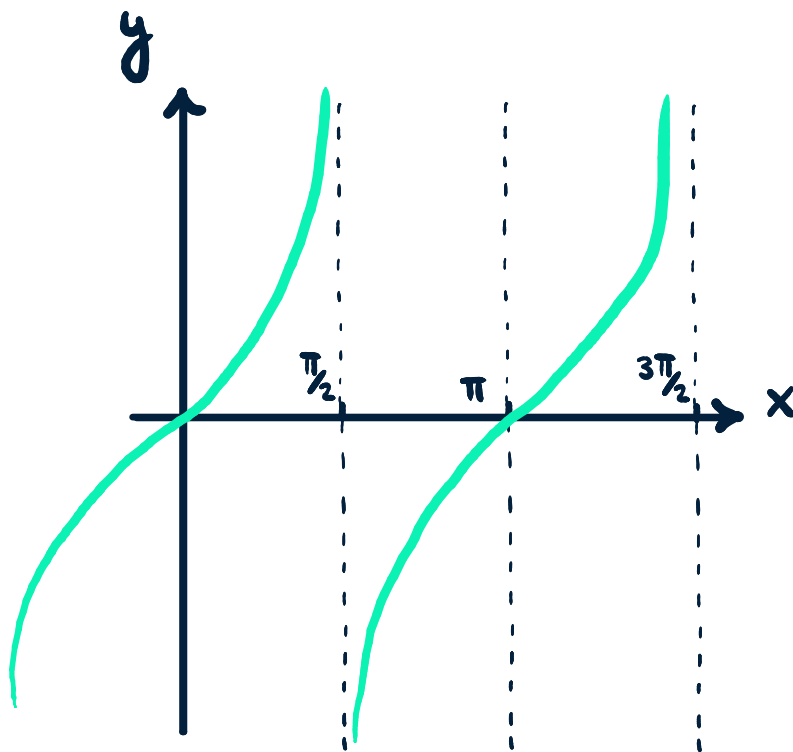
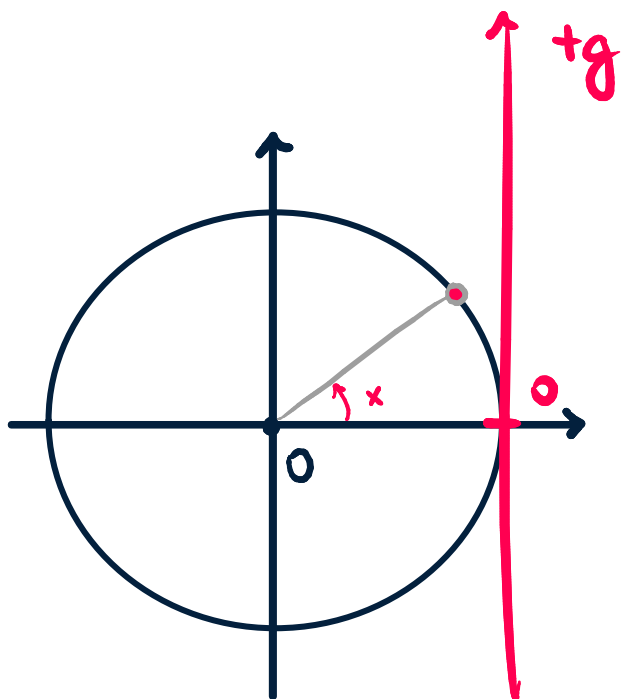


$= \boxed{y}$



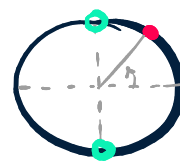
UNIVERSO NARRADO

FUNÇÃO TANGENTE: GRÁFICO

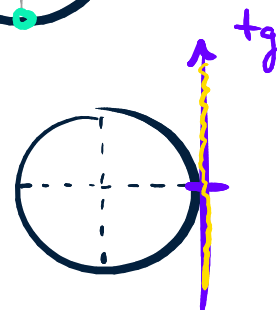


$$f(x) = \tan x$$

• DOMÍNIO : $x \in \mathbb{R} - \{\pi/2 + k\pi\}$



• IMAGEM : $y \in \mathbb{R}$

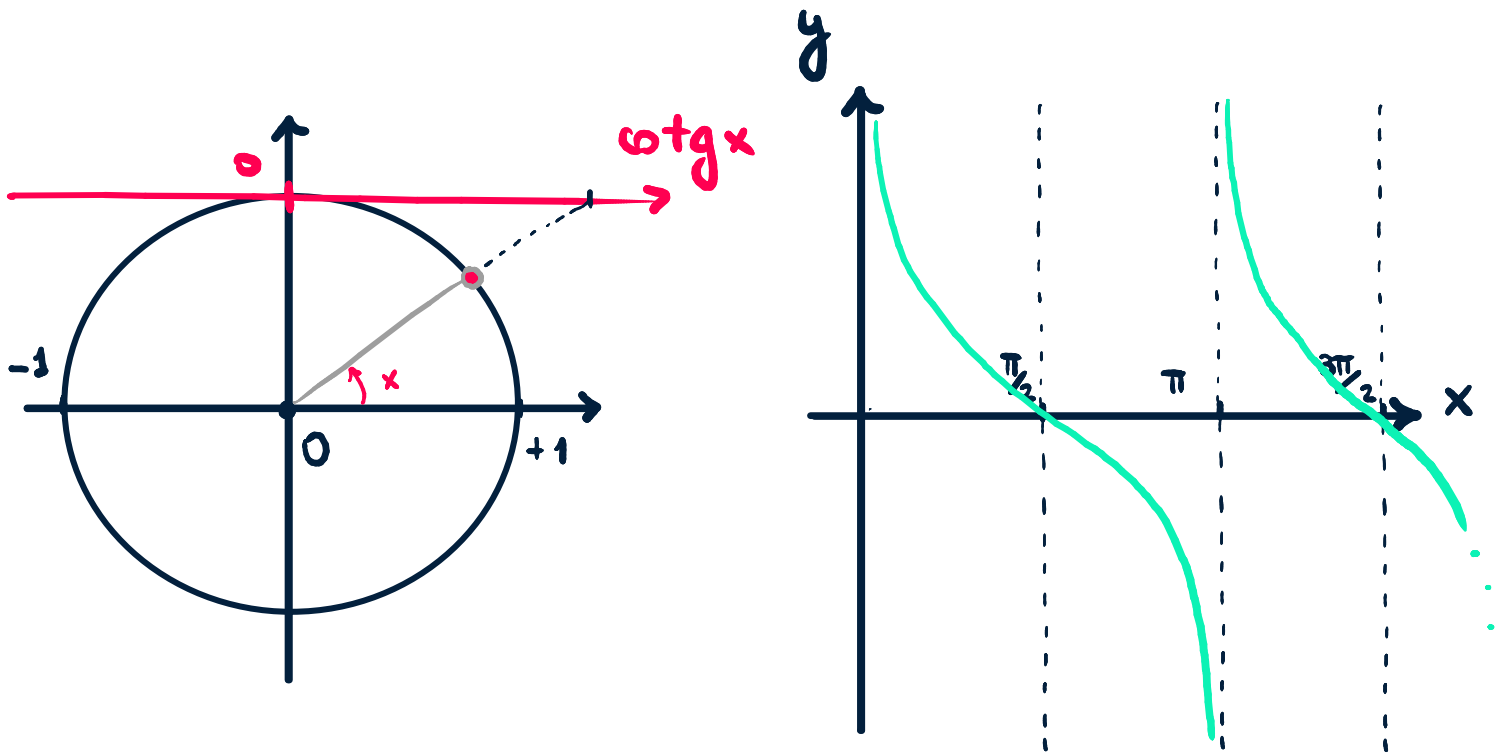


• PERÍODO : π : $\tan(x + \pi) = \tan x$

• PARIDADE : ÍMPAR



FUNÇÃO COTANGENTE.

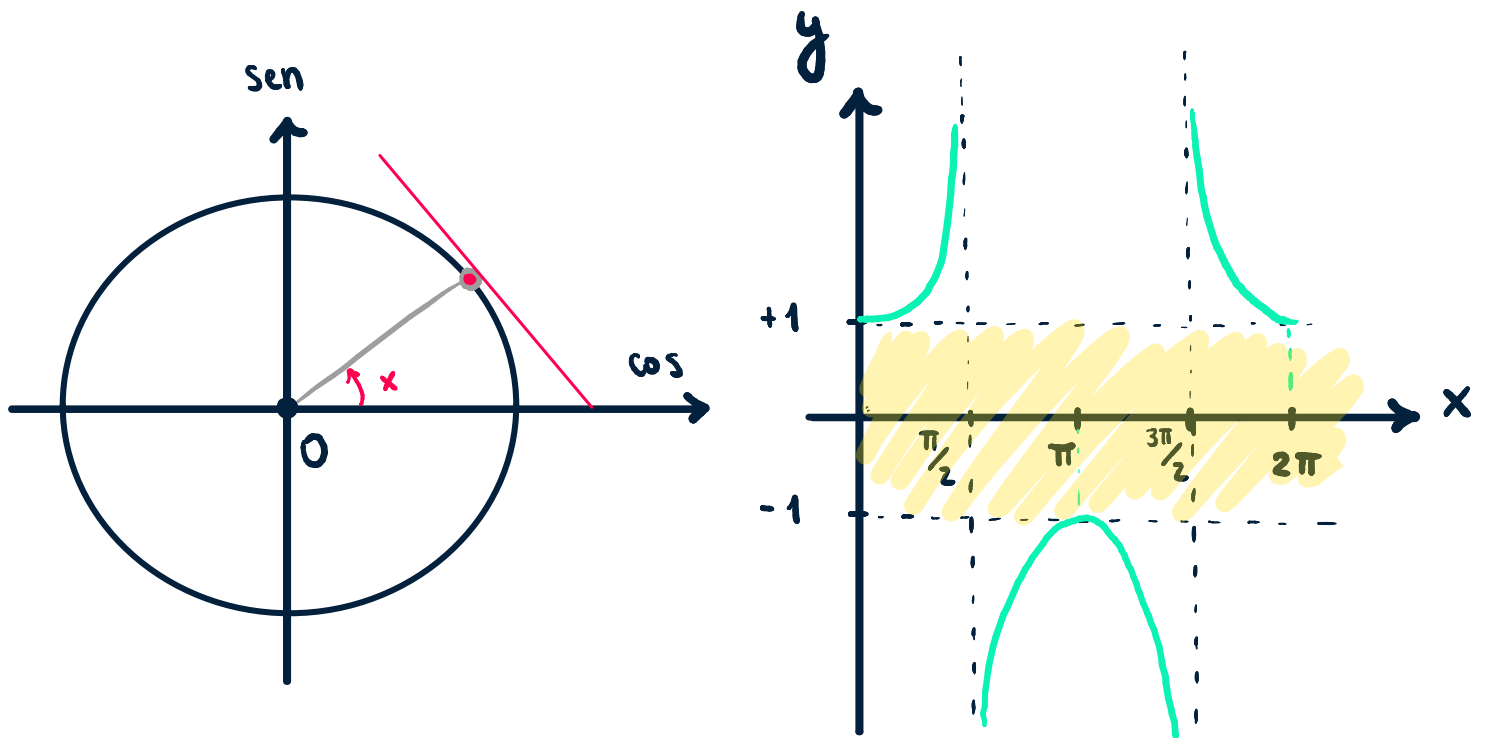


$$f(x) = \cotan x$$

- DOMÍNIO : $x \in \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}$
- IMAGEM: $y \in \mathbb{R}$ [$\text{Im}(f) = \mathbb{R}$]
- PARIDADE: ÍMPAR
- PERÍODO: π : $\cotg(x + \pi) = \cotg x$



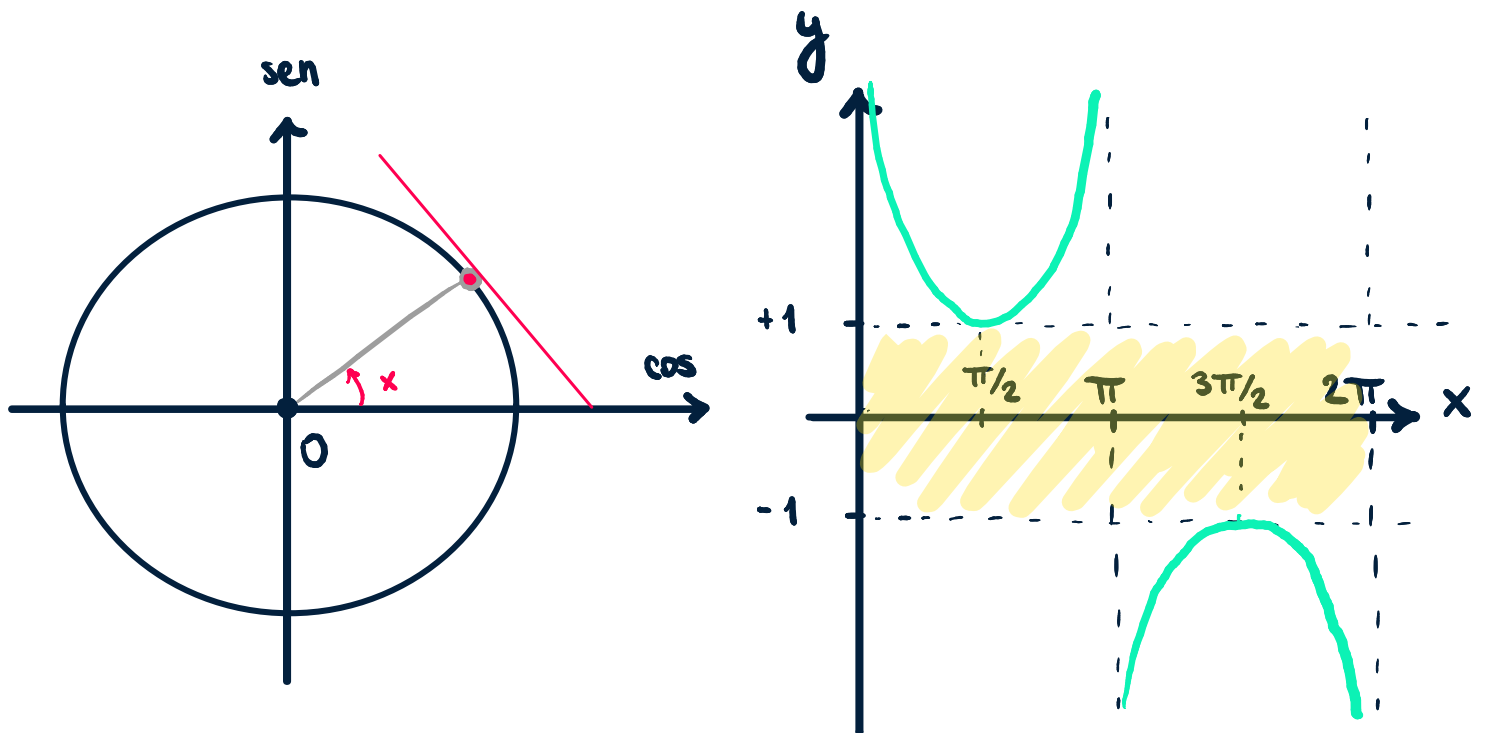
· FUNÇÃO SECANTE ·



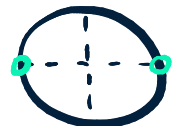
- DOMÍNIO : $x \in \mathbb{R} - \{\pi/2 + k\pi, k \in \mathbb{Z}\}$ 
- IMAGEM: $I(f) = \{y \in \mathbb{R} \mid y \geq +1 \text{ ou } y \leq -1\}$
- PARIDADE: PAR
- PERÍODO : 2π : $\sec(x + 2\pi) = \sec x$



· FUNÇÃO COSSECANTE ·



· DOMÍNIO : $x \in \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}$



· IMAGEM: $\text{Im}(f) = \{y \in \mathbb{R} \mid y \geq +1 \text{ ou } y \leq -1\}$

· PARIDADE: ÍMPAR

· PERÍODO: 2π : $\csc(x + 2\pi) = \csc x$



EXEMPLO 01

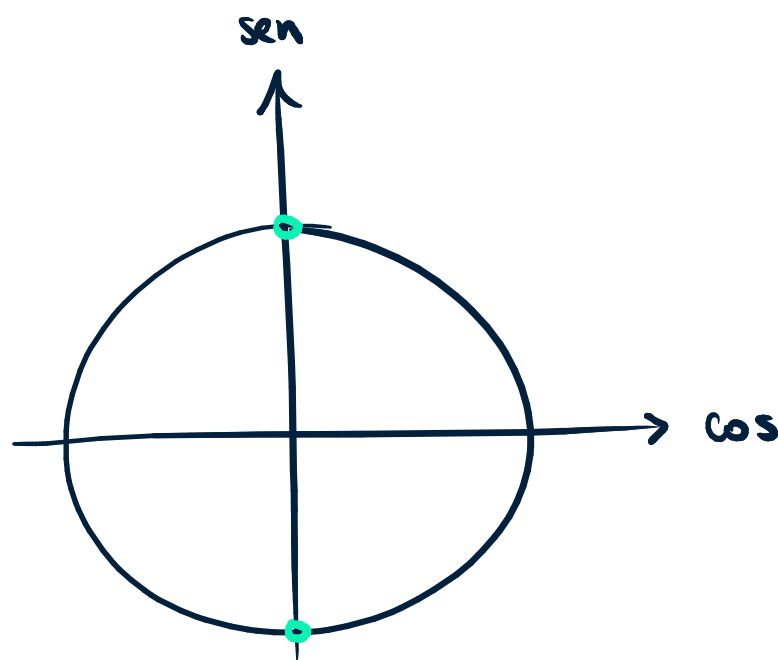
Qual é o domínio da função $f(x) = \tan(x - \pi/6)$?

R :

$$f(x) = \tan(x - \pi/6) = \frac{\text{sen}(x - \pi/6)}{\text{cos}(x - \pi/6)}$$

$\text{cos}(x - \pi/6) \neq 0$

$$x - \pi/6 \neq \pi/2 + k\pi$$



$$x \neq \frac{2\pi}{3} + k\pi$$

$$\text{Domínio} = x \in \mathbb{R} \mid x \neq \left\{ \frac{2\pi}{3} + k\pi \right\}, k \in \mathbb{Z}$$



EXEMPLO 02

Determine o domínio e a imagem da função

$$f(x) = 3 + \sec(3x)$$

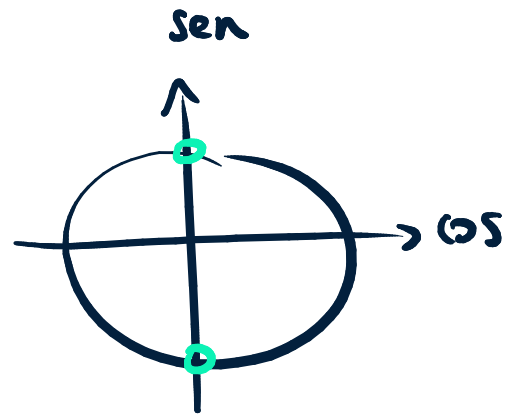
R:

$$f(x) = 3 + \frac{1}{\cos(3x)}$$

(i) $\cos(3x) \neq 0$

$$3x \neq \pi/2 + k\pi$$

$$x \neq \pi/6 + k\pi/3, \quad k \in \mathbb{Z}$$



$$\text{Domínio} = x \in \mathbb{R} \mid x \neq \pi/6 + k\pi/3, \quad k \in \mathbb{Z}$$

(ii) $\text{Im}(f) = y \in \mathbb{R} \mid y \geq 4 \text{ ou } y \leq 2$



TEOREMA

Se uma função f dada por $y = f(x)$ é periódica de período P , então a função

$$g(x) = m + n \cdot f(ax + b)$$

é periódica de período $P/|a|$.

EXEMPLO

$$\text{sen}(x) \rightarrow P = 2\pi$$

$$(i) f(x) = \text{sen}(x/3) \rightarrow a = \frac{1}{3}$$

$$\hookrightarrow P' = \frac{2\pi}{1/3} = 6\pi$$

$$(ii) g(x) = 3 \cos(2x) + 4 \rightarrow \begin{array}{l} \cos x \rightarrow P = 2\pi \\ a = 2 \end{array}$$

$$\hookrightarrow P' = \frac{2\pi}{2} = \pi$$

$$(iii) h(x) = 2 + 3 \tan(x/2 - \pi/5) \rightarrow \begin{array}{l} \tan x \rightarrow P = \pi \\ a = 1/2 \end{array}$$

$$\hookrightarrow P' = \frac{\pi}{1/2} = 2\pi$$



DEMONSTRAÇÃO

Seja $f(x)$ uma função periódica de período P :

$$f(x) = f(x + P) = f(x + 2P) = f(x + 3P) = f(x + kP).$$

$$k \in \mathbb{Z}.$$

$$f(x) = f(x + kP) \quad \downarrow \cdot n$$

$$n \cdot f(x) = n \cdot f(x + kP) \quad \downarrow + m$$

$$n f(\underline{x}) + m = n \cdot f(\underline{x} + kP) + m$$

$$\text{Seja } \underline{x} = ax' + b :$$

$$m + n \cdot f(ax' + b) = m + n \cdot f(\overbrace{ax'} + b + \overbrace{\frac{kPa}{a}})$$

$$m + n \cdot f(ax' + b) = m + n \cdot f\left[\underbrace{a\left(x' + \frac{kP}{a}\right)} + b\right]$$



Seja $\frac{KP}{a} = T$:

$$\underbrace{m + n \cdot f(ax' + b)}_{g(x')} = m + \underbrace{n \cdot f[a(x' + T) + b]}_{g(x' + T)}$$

$$g(x') = g(x' + T)$$

menor: $K = 1$

É periódica de período $T = \frac{KP}{a}$

$$T = \frac{P}{|a|}$$

$$g(x') = m + n \cdot f(ax' + b)$$

$$g(x' + T) = m + n \cdot f[a(x' + T) + b]$$



Sejam $f(x)$ e $g(x)$ funções periódicas de período P_1 e P_2 , respectivamente, com $P_1 \neq P_2$.

TEOREMA

Se $\frac{P_1}{P_2} = \frac{m}{n}$, com m e n inteiros e primos entre si, então as funções $h(x) = f(x) \pm g(x)$ e $t(x) = f(x) \cdot g(x)$ são periódicas de período $np_1 = mp_2$.

EXEMPLO

$$(i) \left. \begin{array}{l} f(x) = \cos(x) \rightarrow P_1 = 2\pi \\ g(x) = \operatorname{tg}(x) \rightarrow P_2 = \pi \end{array} \right\} \quad \frac{P_1}{P_2} = \frac{2\pi}{\pi} = \frac{2}{1}$$

$$h(x) = f(x) \cdot g(x) = \cancel{\cos(x)} \cdot \frac{\cancel{\operatorname{sen}(x)}}{\cancel{\cos(x)}} \quad \frac{P_1}{P_2} = \frac{2}{1} \therefore P' = 2\pi$$

$$(ii) \text{ Calcule o período de } h(x) = \underbrace{\operatorname{sen}(4x)}_1 - \underbrace{\operatorname{tg}(x)}_2$$

$$\left. \begin{array}{l} P_1 = \frac{2\pi}{4} = \frac{\pi}{2} \\ P_2 = \pi \end{array} \right\} \quad \frac{P_1}{P_2} = \frac{\pi/2}{\pi} = \frac{1}{2} \therefore \frac{P_1}{P_2} = \frac{1}{2}$$

$$P' = \pi$$

DEMONSTRAÇÃO

Sejam $f(x)$ e $g(x)$ funções periódicas de período P_1 e P_2 , respectivamente.

Sejam $\alpha(x) = f(x) + g(x)$ e $\beta(x) = f(x) \cdot g(x)$

$$(i) \quad f(x) = f(x + \overbrace{n \cdot P_1}^{\text{inteiro}})$$
$$(+)\quad g(x) = g(x + \underbrace{m \cdot P_2}_{\text{inteiro}})$$

$$\alpha(x) = f(x + n \cdot P_1) + g(x + m \cdot P_2)$$

$$\text{Se } \frac{P_1}{P_2} = \frac{m}{n} \quad \therefore \quad P_1 \cdot n = P_2 \cdot m = T$$

$$\alpha(x) = f(x + n \cdot P_1) + g(x + m \cdot P_2)$$

$$f(x) + g(x) = f(x + T) + g(x + T)$$



$$(ii) \quad f(x) = f(x + n \cdot P_1)$$

$$* \quad g(x) = g(x + m \cdot P_2)$$

$$\beta(x) = f(x + n \cdot P_1) \cdot g(x + m \cdot P_2)$$

$$\text{Se } \frac{P_1}{P_2} = \frac{m}{n} \quad \therefore \quad P_1 \cdot n = P_2 \cdot m = T$$

$$f(x) \cdot g(x) = f(x + T) \cdot g(x + T)$$

$$h(x) = h(x + T)$$

↪ menor

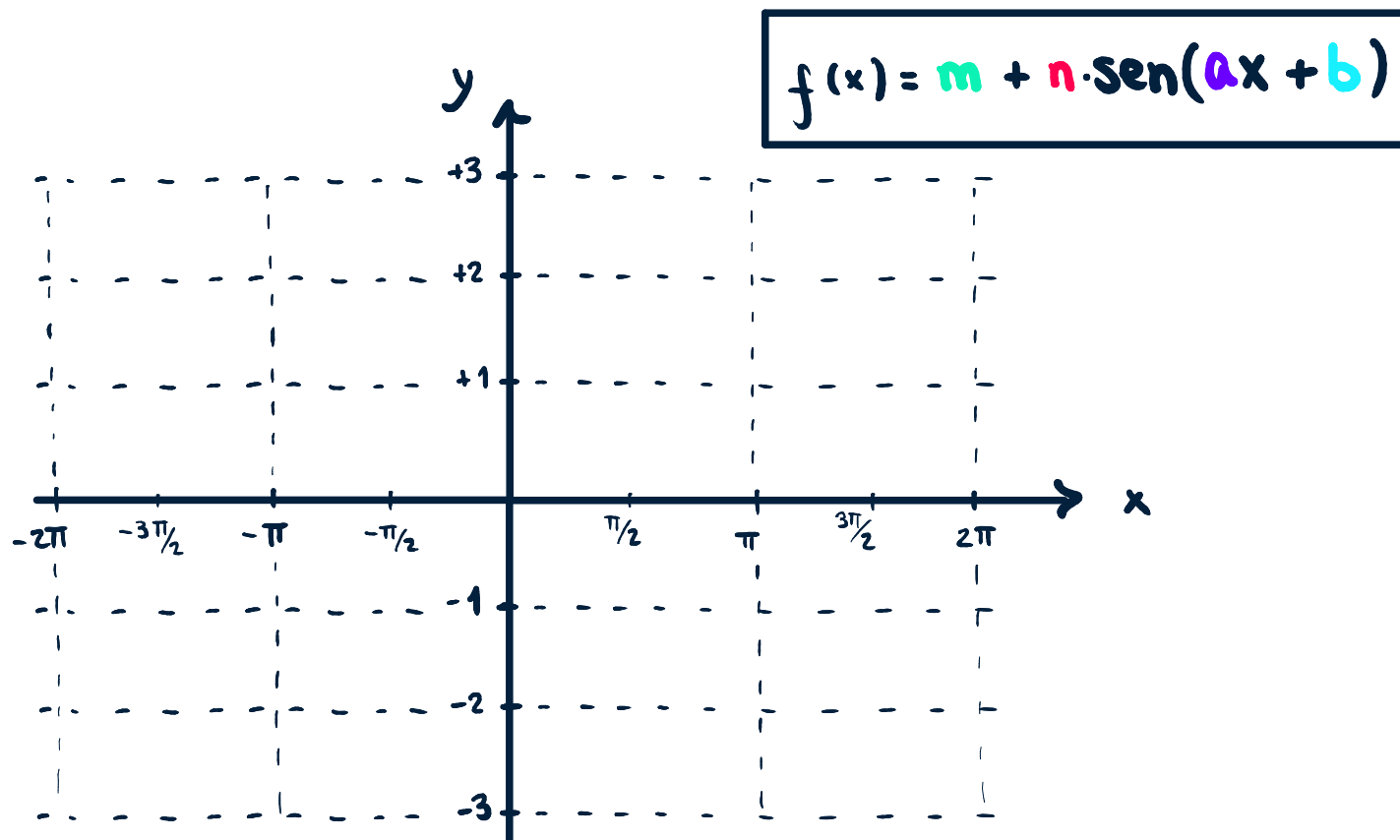


• CONSTRUÇÃO DE GRÁFICOS •

$$f(x) = m + n \cdot \text{sen}(ax + b)$$



$$f(x) = 2 + 3 \cdot \text{sen}(2x + \pi)$$

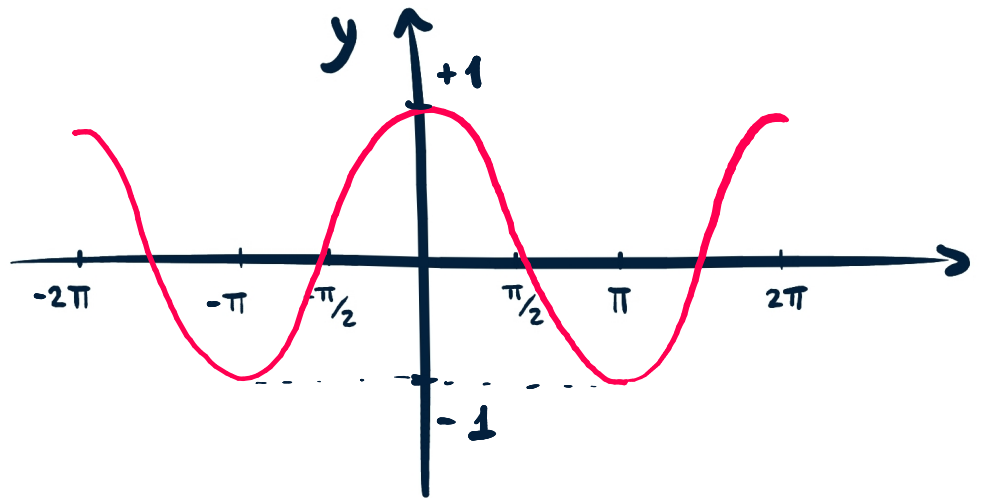


EXEMPLO 01

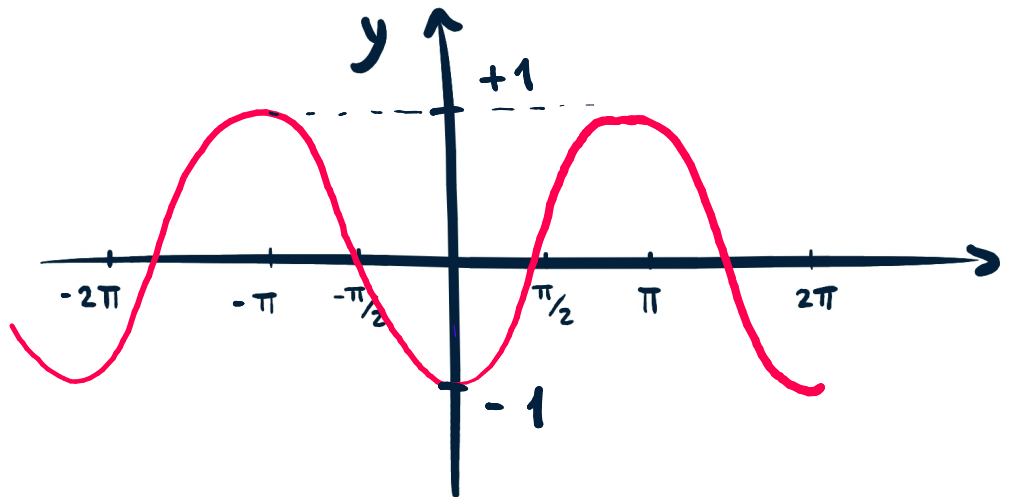
Construa o gráfico da função $f(x) = 2 + \cos(x + \pi)$

R:

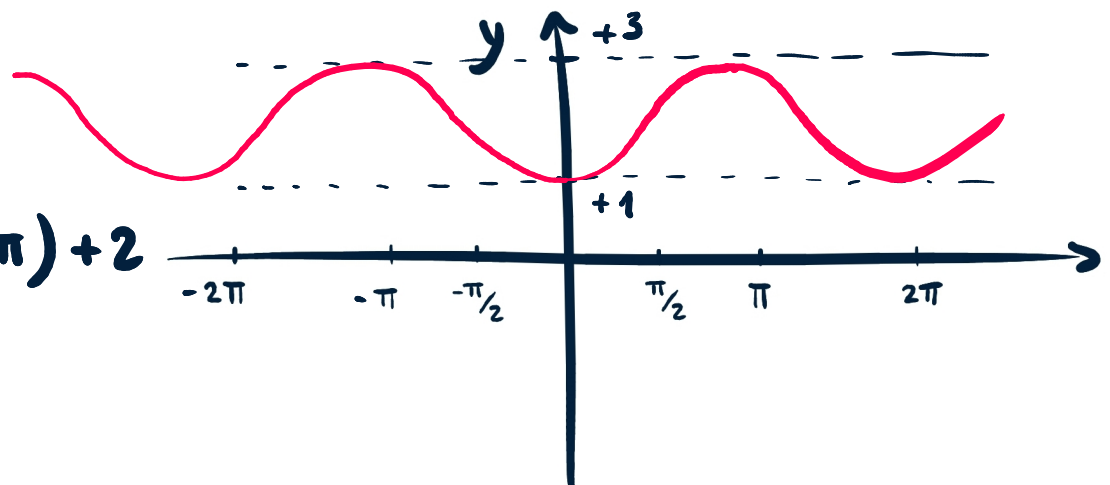
$$g(x) = \cos(x)$$



$$h(x) = \cos(x + \pi)$$



$$f(x) = \cos(x + \pi) + 2$$

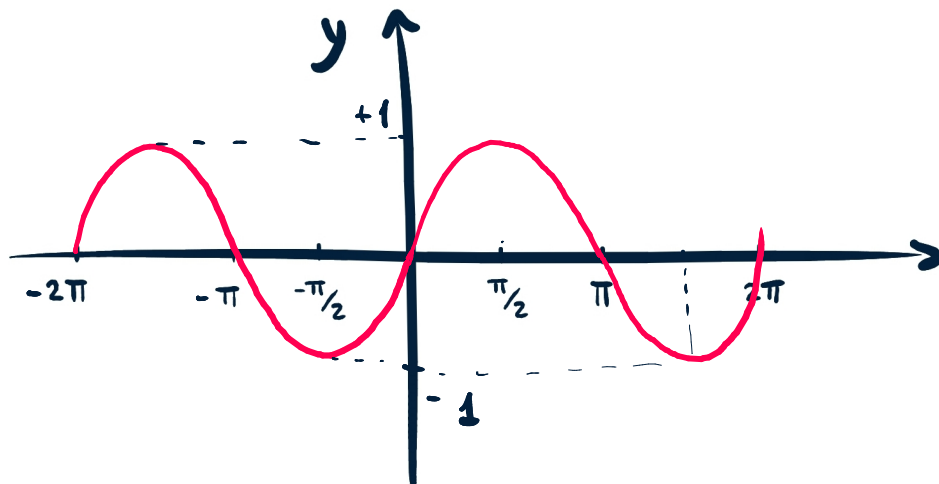


EXEMPLO 02

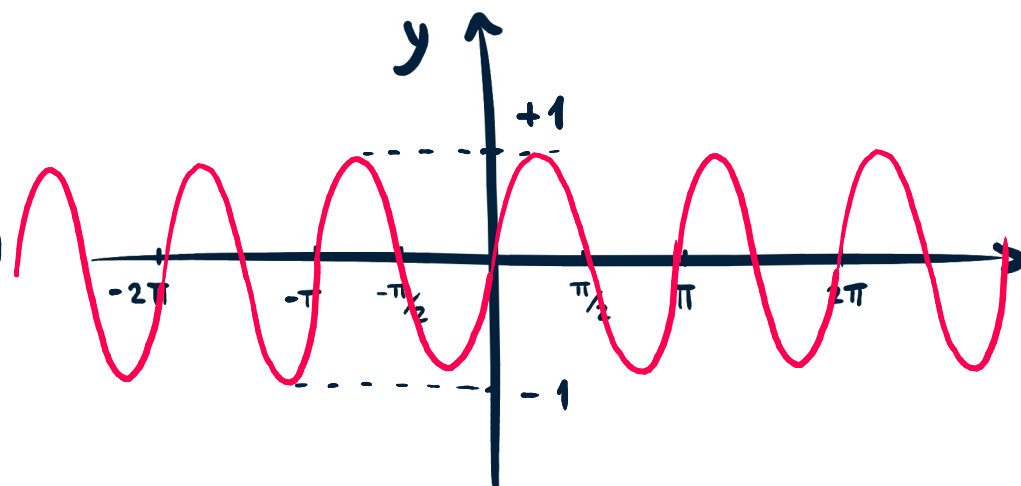
Construa o gráfico da função $f(x) = 1 + \sin(2x)$

R:

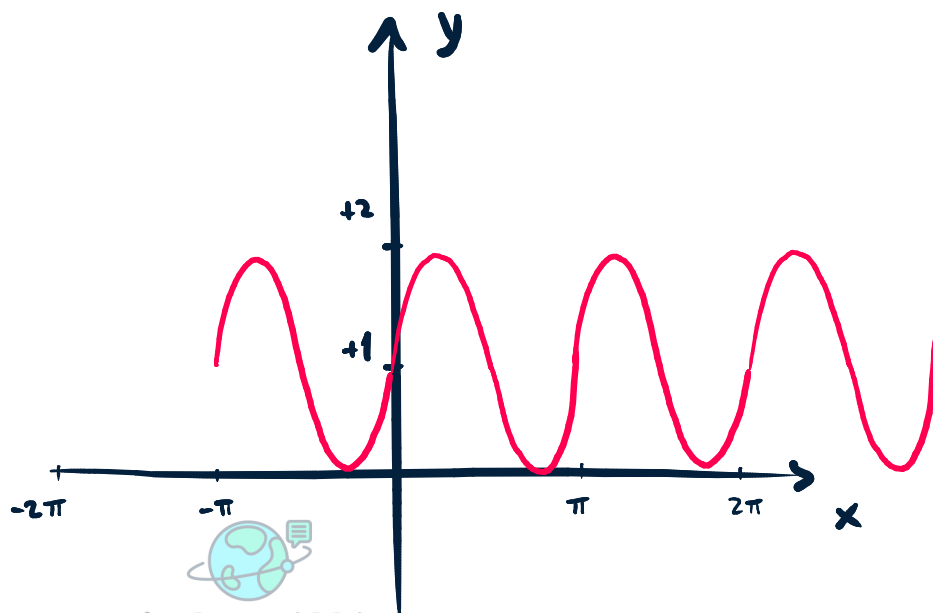
$$g(x) = \sin(x)$$



$$h(x) = \sin(2x)$$



$$f(x) = 1 + \sin(2x)$$

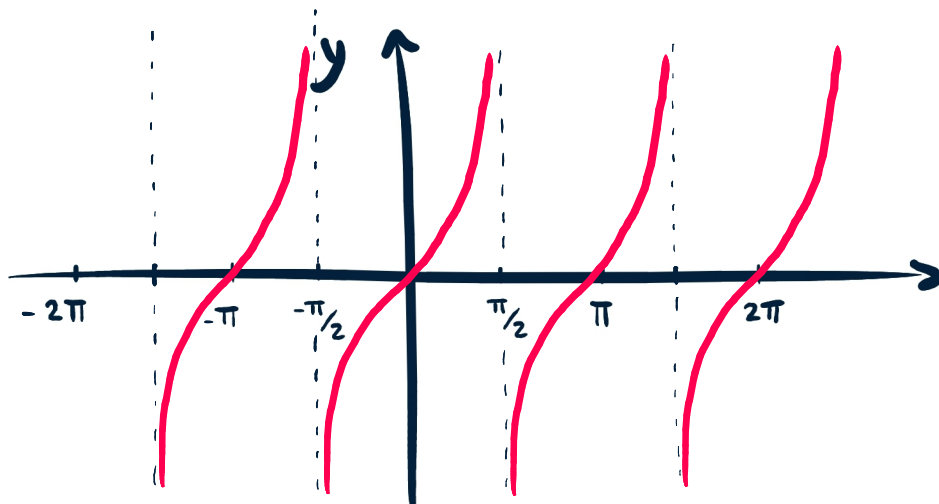


EXEMPLO 03

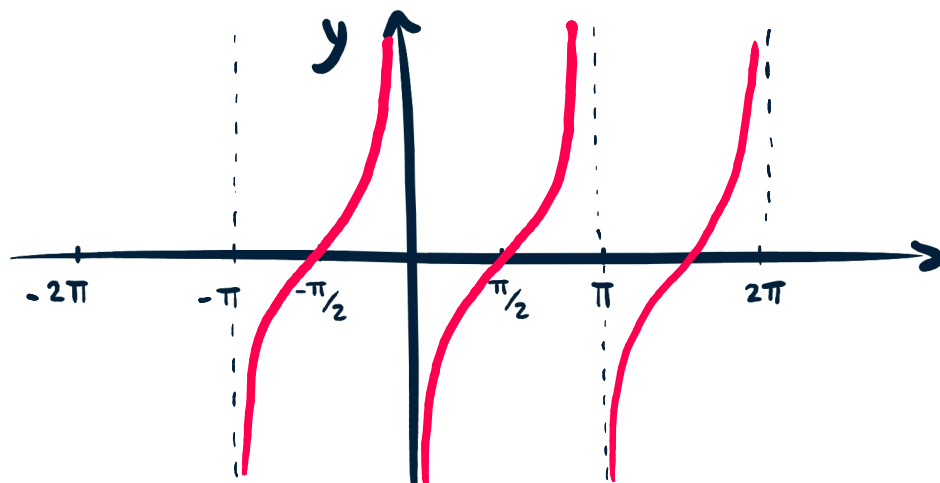
Construa o gráfico da função $f(x) = \operatorname{tg}(x - \pi/2)$

R:

$$g(x) = \operatorname{tg}(x)$$



$$h(x) = \operatorname{tg}(x - \pi/2)$$

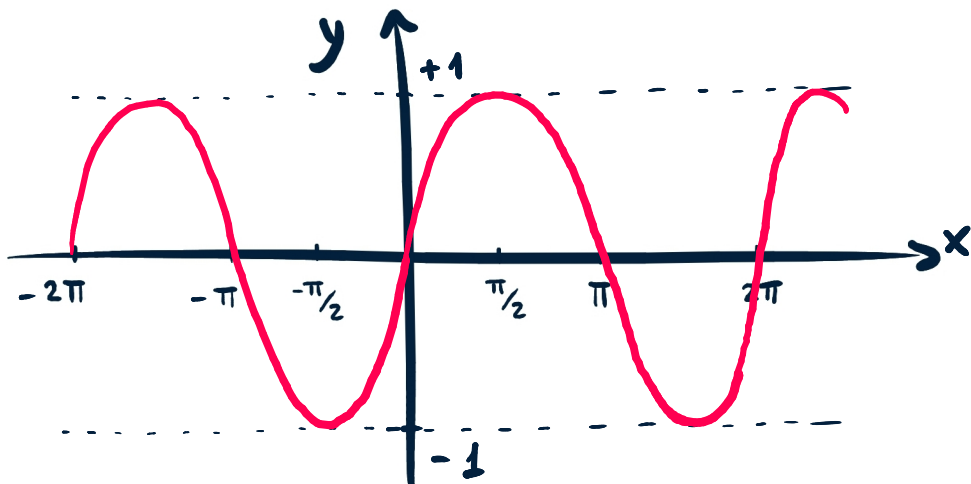


EXEMPLO 04

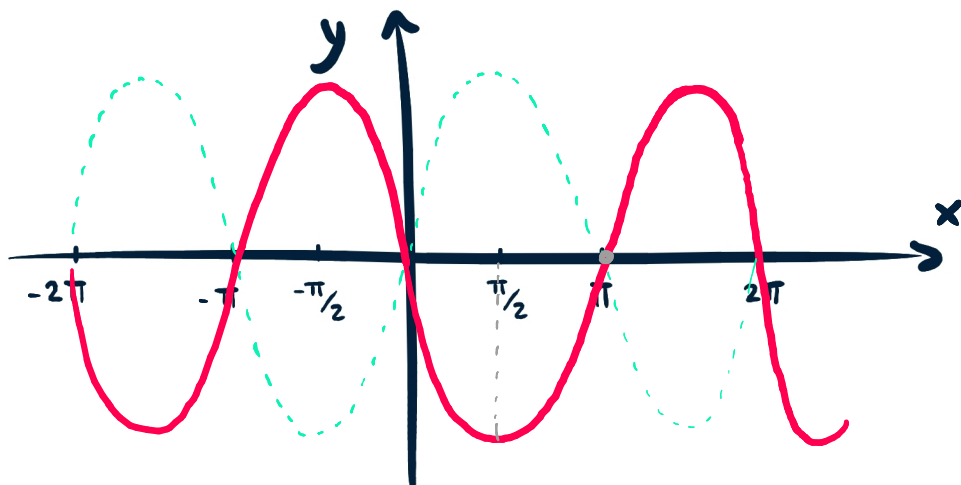
Construa o gráfico da função $f(x) = -\sin(x - \pi/2)$

R:

$$g(x) = \sin x$$

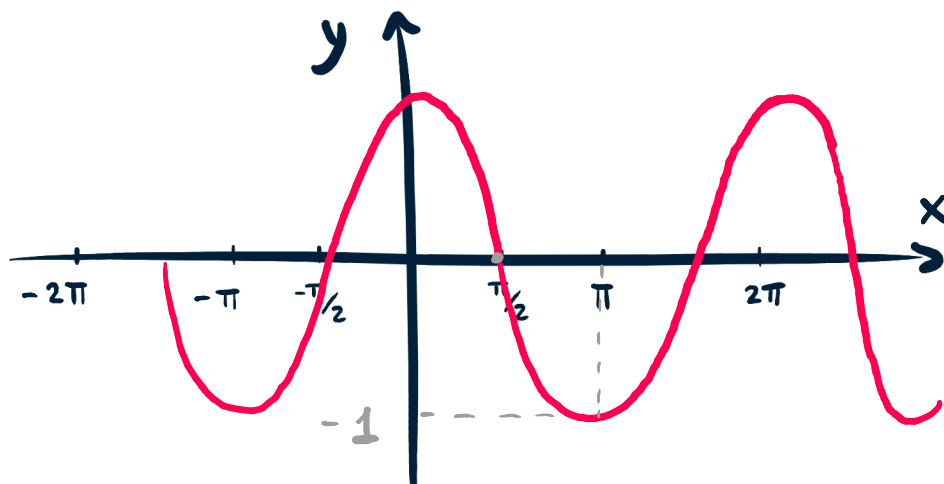


$$h(x) = -\sin x$$



$$f(x) = -\sin(x - \pi/2)$$

→ $\pi/2$

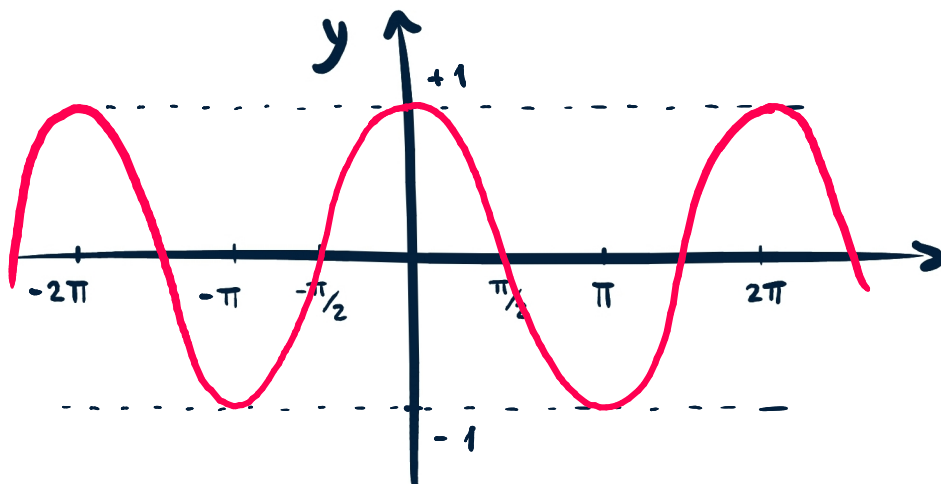


EXEMPLO 05

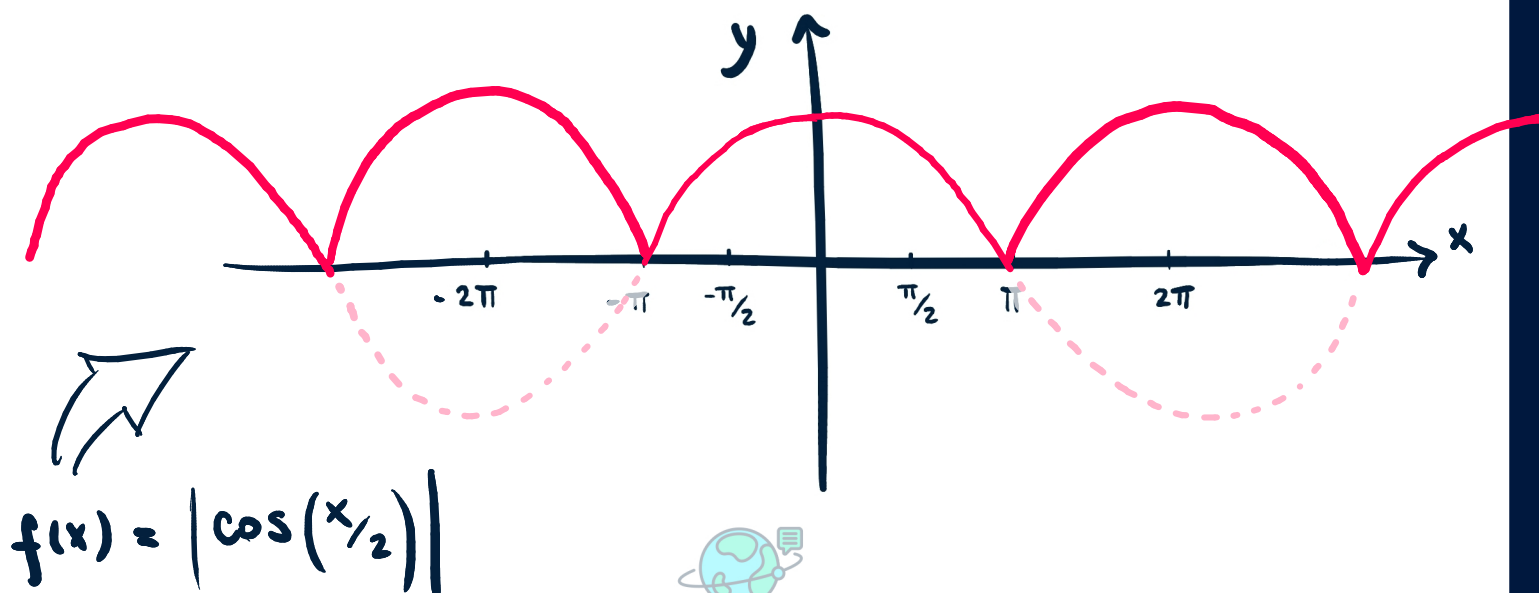
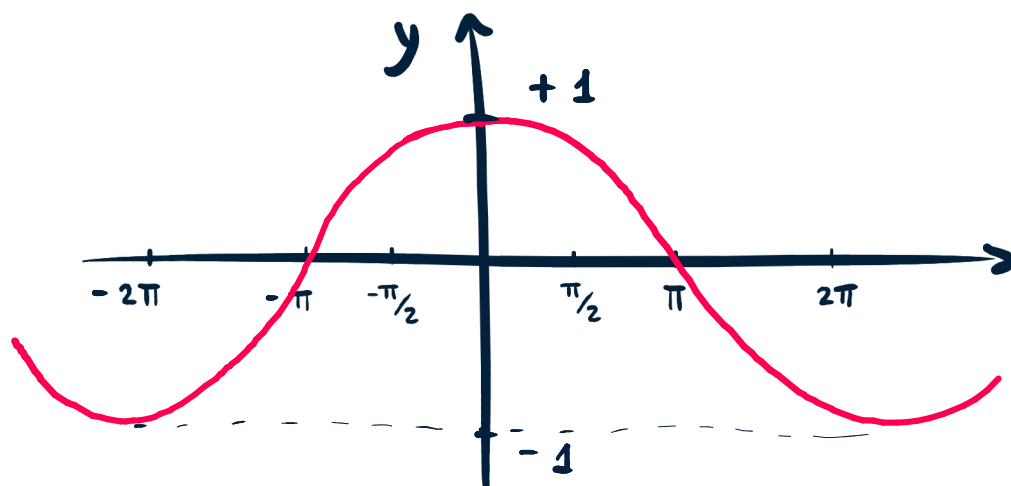
Construa o gráfico da função $f(x) = |\cos(x/2)|$

R:

$$g(x) = \cos(x)$$



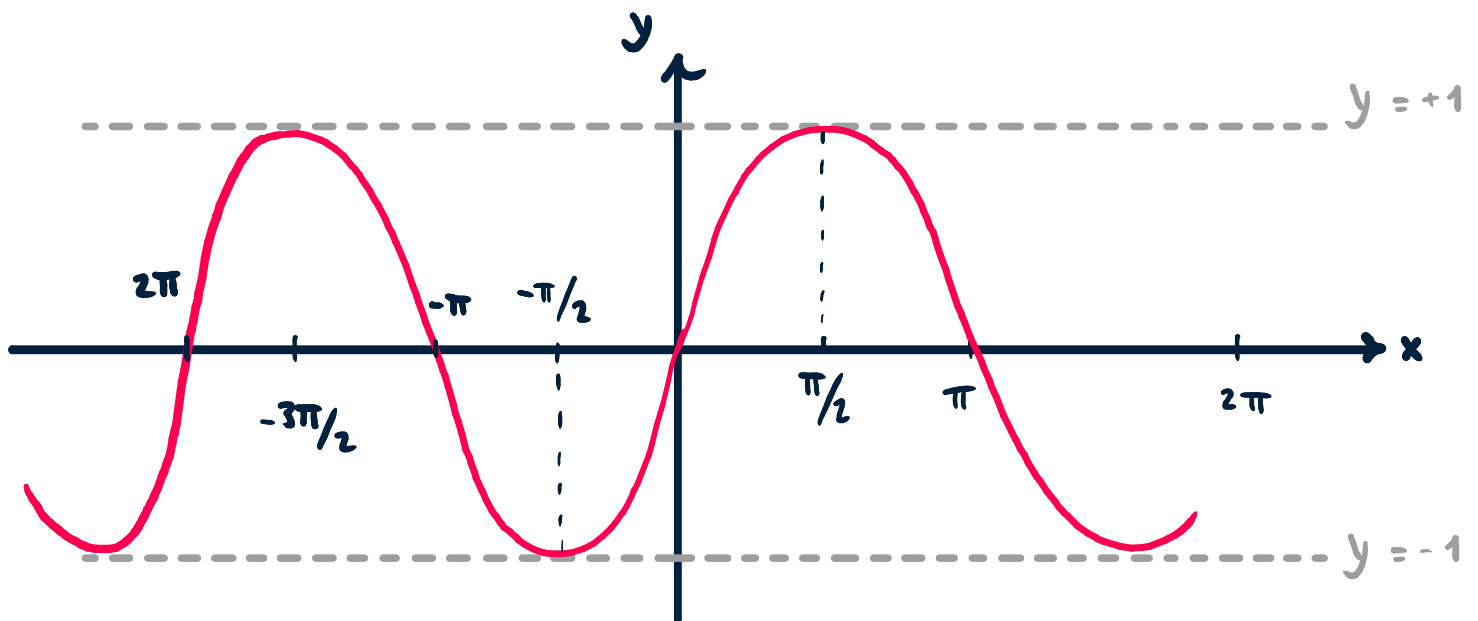
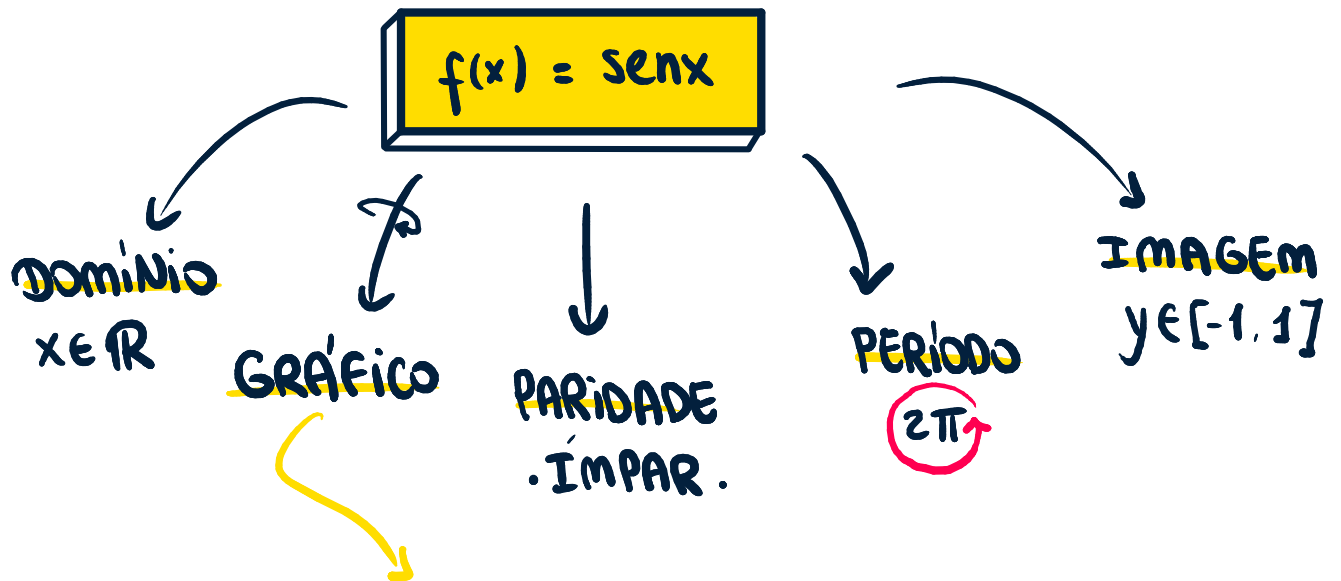
$$h(x) = \cos(x/2)$$



$$f(x) = |\cos(x/2)|$$



RESUMO



$$f(x) = \cos x$$

DOMÍNIO
 $x \in \mathbb{R}$

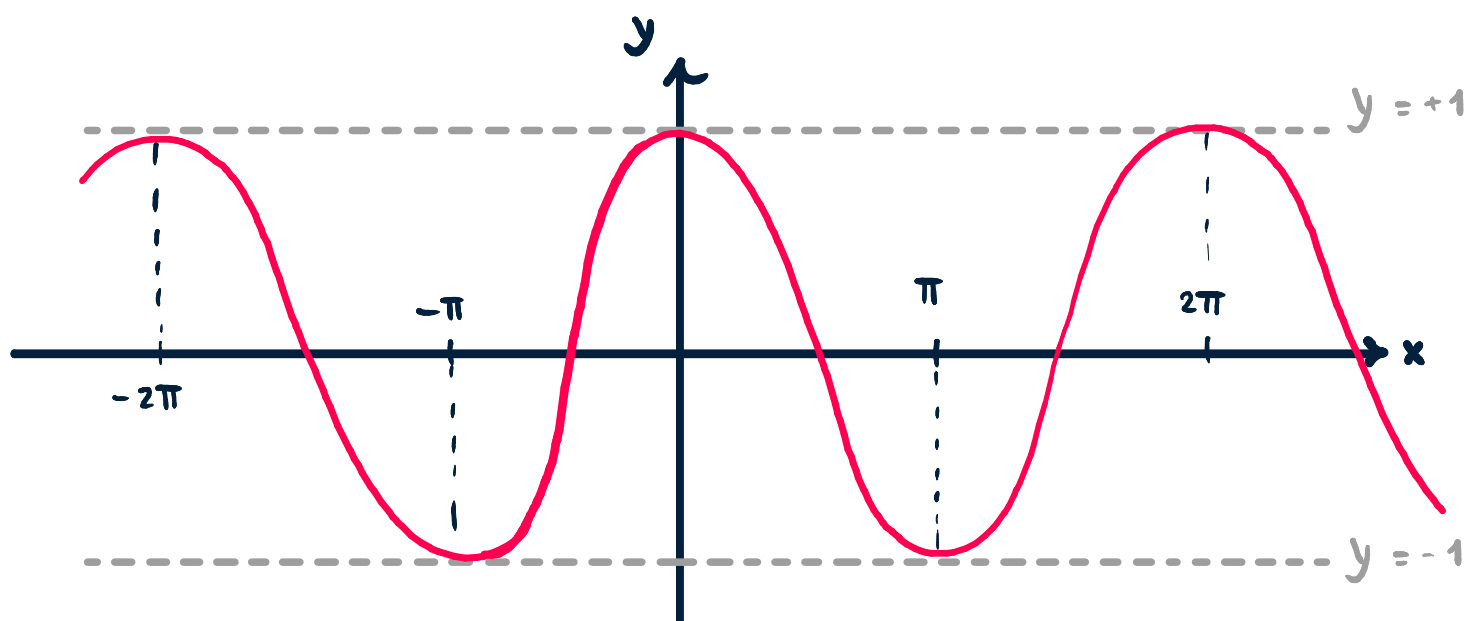
GRÁFICO

PARIDADE
• PAR.

PERÍODO

2π

IMAGEM
 $y \in [-1, 1]$





$$f(x) = \tan x$$

DOMÍNIO
 $x \in \mathbb{R} \mid$
 $x \neq \{\pi/2 + k\pi\}$

GRÁFICO

PARIDADE
· IMPAR.

PERÍODO
 π

IMAGEM
 $y \in \mathbb{R} :$
 $] -\infty, +\infty [$

